4

POTENTIAL & GAUGE

Introduction. When Newton wrote $\boldsymbol{F} = m\ddot{\boldsymbol{x}}$ he imposed no significant general constraint on the design of the force law $\boldsymbol{F}(\boldsymbol{x}, t)$. God, however, appears to have special affection for *conservative* forces—those (a subset of zero measure within the set of all conceivable possibilities) that conform to the condition

$$\boldsymbol{\nabla}\times\boldsymbol{F} = \boldsymbol{0}$$

—those, in other words, that can be considered to derive from a scalar potential:

$$\boldsymbol{F} = -\boldsymbol{\nabla}U \tag{357}$$

Only in such cases is it

- possible to speak of energy conservation
- possible to construct a Lagrangian $L = T - U$
- possible to construct a Hamiltonian $H = T + U$
- possible to quantize.

It is, we remind ourselves, the potential U—not the force $\boldsymbol{F}$—that appears in the Schrödinger equation ... which is rather remarkable, for U has the lesser claim to direct physicality: if U "does the job" (by which I mean: if U reproduces $\boldsymbol{F}$) then so also does

$$U \equiv U + \text{constant} \tag{358}$$

where "constant" is vivid writing that somewhat overstates the case: we require only that $\boldsymbol{\nabla}\cdot(\text{constant}) = \boldsymbol{0}$, which disallows $\boldsymbol{x}$-dependence but does not disallow t-dependence.

At (357) a "spook" has intruded into mechanics—a device which we are content to welcome into (and in fact can hardly exclude from) our computational lives . . . but which, in view of (358), cannot be allowed to appear nakedly in our final results. The adjustment

$$U \longrightarrow U = U + \text{constant}$$

provides the simplest instance of what has come in relatively recent times to be called a "gauge transformation."[206] For obvious reasons we require of such physical statements as may contain U that they be *gauge-invariant*. To say the same thing another way: It is permissible to write (say)

$$E = \tfrac{1}{2}m\dot{x}^2 + U(\boldsymbol{x})$$

in the midst of a theoretical argument, but it would be pointless to go to the stockroom in quest of a "U-meter": the best we could do would be to obtain a "potentiometer" . . . that has *two* testleads and measures

$$\Delta U = U(\boldsymbol{x}) - U(\boldsymbol{x}_0) \quad : \quad \text{gauge-invariant}$$

Or a "differential potentiometer," that measures $\boldsymbol{\nabla} U$.

Moving deeper into mechanics, we encounter the Lagrangian $L(q, \dot{q}, t)$, which (though seldom described in such terms) must itself be a kind of "potential"—a "spook"—since susceptible to gauge transformations of the form

$$L(q, \dot{q}, t) \longrightarrow L(q, \dot{q}, t) + \tfrac{d}{dt}(\text{any function of } q \text{ and } t)$$

—the point here being that if L and L are so related then they give rise to identical equations of motion.

We encountered the scalar potential already when at (17) we had occasion to write

$$\boldsymbol{E} = -\boldsymbol{\nabla}\varphi \quad : \quad \text{invariant under} \quad \varphi \longrightarrow \varphi = \varphi + \text{constant} \tag{359.1}$$

and to observe that it is characteristic of the structure of electro*static* fields that

$$\boldsymbol{\nabla}\times\boldsymbol{E} = \boldsymbol{0} \tag{359.2}$$

In a parallel discussion of *magnetostatic* fields we were led at (92) to the "vector potential;"[207] *i.e.*, to the observation that if we write

[206] The terminology is due, I have read, to Hermann Weyl (the founding father of what became "gauge field theory"), who reportedly had in mind the "gauge" of railway tracks.

[207] The vector potential first appears (∼1835) in work of F. E. Neumann (1798–1895) concerned with the mechanical interaction of current-carrying wires (Ampere's law: see page 58). Maxwell (1831–1879) came independently to the same idea at a much later date, and from a different direction (Faraday's law). Neumann, by the way, was a close associate of Jacobi (1804–1851) from 1833 until the younger man's death, and was the teacher of many of the greatest figures in 19th Century German physics.

$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} \quad : \quad \text{invariant under} \quad \boldsymbol{A} \longrightarrow \boldsymbol{A} = \boldsymbol{A} + \boldsymbol{\nabla}\chi \tag{359.3}$$

then

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \tag{359.4}$$

is rendered automatic.

So important is the role played by scalar/vector potentials in all vector field theories—in fluid dynamics, for example, but especially in electrodynamics—that in this chapter I interrupt the flow of the narrative to indicate how those concepts fit within the framework of the manifestly covariant theory of the electromagnetic field. The ideas presented here will be central to all of our subsequent work.

1. How potentials come into play: Helmholtz' decomposition theorem. In three dimensions, a vector field $\boldsymbol{V}(\boldsymbol{x})$ is said to be

- "irrotational" if and only if $\boldsymbol{\nabla}\times\boldsymbol{V} = \boldsymbol{0}$
- "solenoidal" if and only if $\boldsymbol{\nabla}\cdot\boldsymbol{V} = 0$.

Helmholtz (and later but independently also Maxwell) showed that *every* vector field can be resolved[208]

$$\boldsymbol{V}(\boldsymbol{x}) = \big\{\text{irrotational part } \boldsymbol{I}(\boldsymbol{x})\big\} + \big\{\text{solenoidal part } \boldsymbol{S}(\boldsymbol{x})\big\} \tag{360}$$

Drawing now upon the (unproven) converse of (6) we conclude that $\boldsymbol{I}$ can be considered to arise by

$$\boldsymbol{I} = \boldsymbol{\nabla}\psi$$

from a *scalar potential* ψ, and that $\boldsymbol{S}$ can be considered to arise by

$$\boldsymbol{S} = \boldsymbol{\nabla}\times\boldsymbol{\psi}$$

from a *vector potential* $\boldsymbol{\psi}$. Every vector field $\boldsymbol{V}$ can therefore be displayed

$$\boldsymbol{V} = \boldsymbol{\nabla}\psi + \boldsymbol{\nabla}\times\boldsymbol{\psi} = \text{gradient} + \text{curl} \tag{361}$$

but that display is *non-unique*, since the *potentials are determined only to within gauge transformations*

$$\left.\begin{aligned} \psi &\longrightarrow \psi = \psi + \text{arbitrary constant} \\ \boldsymbol{\psi} &\longrightarrow \boldsymbol{\psi} = \boldsymbol{\psi} + \boldsymbol{\nabla}(\text{arbitrary scalar field}) \end{aligned}\right\} \tag{362}$$

Since susceptible to gauge transformation, the potentials ψ and $\boldsymbol{\psi}$ are released from adherence to such boundary/symmetry/transformation properties as—in specific applications—typically pertain to the "physical" fields $\boldsymbol{V}$.

[208] For proof see R. B. McQuistan, *Scalar & Vector fields: A Physical Interpretation* (1965), page 261.

It is not obvious that the replacement of three objects (the components of the vector $\boldsymbol{V}$) by four (ψ and the components of $\boldsymbol{\psi}$) represents an advance. But in applications it is invariably the case that Helmholtz decomposition (360) serves to *clarify the essential structure* of the theory in question, and is often the case that by exploiting gauge freedom one can *simplify both the formulation of the theory and many of the attendant computations*. The electrodynamical application will serve to illustrate both of those advantages.

Helmholtz decomposition provides the simplest instance of the vastly more general "Hodge decomposition," which (though not usually phrased in such terms) can be considered to pertain to completely antisymmetric tensors of arbitrary rank, inscribed on N-dimensional manifolds of almost arbitrary topology.[209]

2. Application to Maxwellian electrodynamics. Look again to the pair of Maxwell equations that make no reference to source activity:

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0 \tag{65.2}$$
$$\boldsymbol{\nabla}\times\boldsymbol{E} + \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{B} = \mathbf{0} \tag{65.4}$$

The former asserts that *magnetic fields—not only in the static case, but also dynamically—are solenoidal*, so can be written

$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} \tag{363.1}$$

Returning with this information to (65.4) we obtain $\boldsymbol{\nabla}\times\big\{\boldsymbol{E} + \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}\big\} = \mathbf{0}$, according to which $\boldsymbol{E} + \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}$ is irrotational, so can be expressed $-\boldsymbol{\nabla}\varphi$, giving

$$\begin{aligned}\boldsymbol{E} &= -\boldsymbol{\nabla}\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A} \\ &= \big\{\text{irrotational component arising from charges}\big\} \\ &\qquad\qquad + \big\{\text{component generated by Faraday induction}\big\} \\ &\downarrow \\ &= -\boldsymbol{\nabla}\varphi \quad \text{in the static case}\end{aligned} \tag{363.2}$$

It was, by the way, to place himself in position to write $\boldsymbol{E}_{\text{Faraday}} = -\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}$ that Maxwell was motivated[207] to re*invent* the vector potential.

The construction (363.1) of $\boldsymbol{B}$ is invariant under $\boldsymbol{A} \longrightarrow \boldsymbol{A} = \boldsymbol{A} - \boldsymbol{\nabla}\chi$. But that adjustment sends

$$\boldsymbol{E} = -\boldsymbol{\nabla}\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A} \quad\longrightarrow\quad \begin{aligned}\boldsymbol{E} &= -\boldsymbol{\nabla}\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}(\boldsymbol{A} + \boldsymbol{\nabla}\chi) \\ &= -\boldsymbol{\nabla}\big\{\varphi + \tfrac{1}{c}\tfrac{\partial}{\partial t}\chi\big\} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A}\end{aligned}$$

[209] See H. Flanders, *Differential Forms, with Applications to the Physical Sciences* (1963), page 138.

and that observation motivates us to write $\varphi \equiv \varphi + \frac{1}{c}\frac{\partial}{\partial t}\chi$. To summarize: the equations (363) are invariant under

$$\left.\begin{aligned} \varphi &\longrightarrow \varphi = \varphi + \tfrac{1}{c}\tfrac{\partial}{\partial t}\chi \\ \boldsymbol{A} &\longrightarrow \boldsymbol{A} = \boldsymbol{A} - \boldsymbol{\nabla}\chi \end{aligned}\right\} \tag{364}$$

where χ is an arbitrary scalar field, and where we can look upon the first adjustment as a forced implication of the second.

The source-independent Maxwell equations (65.2) and (65.4) have—by the introduction (363) of the scalar/vector potentials—been rendered *automatic*. We need concern ourselves, therefore, only with the sourcey Maxwell equations

$$\boldsymbol{\nabla}\!\cdot\!\boldsymbol{E} = \rho \tag{65.1}$$
$$\boldsymbol{\nabla}\!\times\!\boldsymbol{B} - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{E} = \tfrac{1}{c}\boldsymbol{j} \tag{65.3}$$

which, when expressed in terms of the potentials, become a *pair* of *second order* partial differential equations:

$$\begin{aligned} \boldsymbol{\nabla}\!\cdot\!\big\{-\boldsymbol{\nabla}\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A}\big\} &= \rho \\ \boldsymbol{\nabla}\!\times\!(\boldsymbol{\nabla}\!\times\!\boldsymbol{A}) - \tfrac{1}{c}\tfrac{\partial}{\partial t}\big\{-\boldsymbol{\nabla}\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A}\big\} &= \tfrac{1}{c}\boldsymbol{j} \end{aligned}$$

These, after simplification[210] and reorganization, can be rendered

$$\begin{aligned} -\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{\nabla}\!\cdot\!\boldsymbol{A} - \nabla^2\varphi &= \rho \\ \big[\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2 - \nabla^2\big]\boldsymbol{A} + \boldsymbol{\nabla}\big\{\tfrac{1}{c}\tfrac{\partial}{\partial t}\varphi + \boldsymbol{\nabla}\!\cdot\!\boldsymbol{A}\big\} &= \tfrac{1}{c}\boldsymbol{j} \end{aligned}$$

or again but more symmetrically (add/subtract a term in the first equation)

$$\left.\begin{aligned} \big[\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2 - \nabla^2\big]\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\big\{\tfrac{1}{c}\tfrac{\partial}{\partial t}\varphi + \boldsymbol{\nabla}\!\cdot\!\boldsymbol{A}\big\} &= \rho \\ \big[\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2 - \nabla^2\big]\boldsymbol{A} + \quad \boldsymbol{\nabla}\big\{\tfrac{1}{c}\tfrac{\partial}{\partial t}\varphi + \boldsymbol{\nabla}\!\cdot\!\boldsymbol{A}\big\} &= \tfrac{1}{c}\boldsymbol{j} \end{aligned}\right\} \tag{365.1}$$

The field equations (365) are *gauge-invariant*, which is to say: under the substitutional adjustment

$$\begin{aligned} \varphi &\longmapsto \varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\chi \\ \boldsymbol{A} &\longmapsto \boldsymbol{A} + \boldsymbol{\nabla}\chi \end{aligned}$$

they go over into

$$\left.\begin{aligned} \big[\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2 - \nabla^2\big]\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\big\{\tfrac{1}{c}\tfrac{\partial}{\partial t}\varphi + \boldsymbol{\nabla}\!\cdot\!\boldsymbol{A}\big\} &= \rho \\ \big[\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2 - \nabla^2\big]\boldsymbol{A} + \quad \boldsymbol{\nabla}\big\{\tfrac{1}{c}\tfrac{\partial}{\partial t}\varphi + \boldsymbol{\nabla}\!\cdot\!\boldsymbol{A}\big\} &= \tfrac{1}{c}\boldsymbol{j} \end{aligned}\right\} \tag{365.2}$$

[210] Recall the identity $\boldsymbol{\nabla}\!\times\!(\boldsymbol{\nabla}\!\times\!\boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\!\cdot\!\boldsymbol{A}) - \nabla^2\boldsymbol{A}$, of which we made use already on page 54.

because all the χ-terms cancel. Gauge freedom can be used to render (365.2) simpler (or, for that matter, more complicated) than (365.1). For example: from $\nabla\cdot\boldsymbol{A} = \nabla\cdot\boldsymbol{A} - \nabla^2\chi$ we learn that if χ is taken to be any solution of

$$\nabla^2\chi = \nabla\cdot\boldsymbol{A}$$

then $\boldsymbol{A}$ satisfies the

$$\text{COULOMB GAUGE CONDITION:}\quad \nabla\cdot\boldsymbol{A} = 0$$

and equations (365.2) become

$$\begin{aligned} \nabla^2\varphi &= -\rho \\ \big[\big(\tfrac{1}{c}\tfrac{\partial}{\partial t}\big)^2 - \nabla^2\big]\boldsymbol{A} &= \tfrac{1}{c}\boldsymbol{j} - \nabla\big\{\tfrac{1}{c}\tfrac{\partial}{\partial t}\varphi\big\} \end{aligned}$$

└formally a kind of "current"

The Coulomb gauge is also known as the "radiation" or "transverse gauge." For discussion see §6.3 in J. D. Jackson's *Classical Electrodynamics* (3^{rd} edition 1999). Of much more general importance is the

$$\text{LORENTZ GAUGE CONDITION:}\quad \tfrac{1}{c}\tfrac{\partial}{\partial t}\varphi + \nabla\cdot\boldsymbol{A} = 0 \tag{366}$$

which arises from taking χ to be any solution of

$$\Box\chi = -\big\{\tfrac{1}{c}\tfrac{\partial}{\partial t}\varphi + \nabla\cdot\boldsymbol{A}\big\}$$

and which brings (365.2) to the strikingly simple form

$$\left.\begin{aligned} \Box\varphi &= \rho \\ \Box\boldsymbol{A} &= \tfrac{1}{c}\boldsymbol{j} \end{aligned}\right\} \tag{367}$$

HISTORICAL REMARK: I have been informed by David Griffiths (who learned from J. D. Jackson, while on sabbatical at Berkeley) that (366) first appears in the work (1867)not of H. A. Lorentz (Dutch, 1853–1928) but of L. V. Lorenz (Danish, 1829–1891), so should —in violation of universal practice—be called the "Lorenz gauge condition" (no "t"). For the fascinating historical details see J. D. Jackson & L. B. Okun, "Historical roots of gauge invariance," RMP **73**, 6653 (2001). My own recent effort to discover the facts of the matter took me to the *Dictionary of Scientific Biography* (1973), where I was reminded that the Lorentz article—by Russell McCormmach, an eminent historian of physics who was once my Reed College classmate—provides a splendid short account of the confused state of electrodynamics when Lorentz entered upon the scene. Theories by Weber, Neumann, Riemann, Lorenz and—almost lost in the crowd—Maxwell were then in lively competition. McCormmach makes clear the insightful audacity that Lorentz displayed when he embraced a theory that assigned a central place to a perplexing notion (the field concept) and that declined to address a question that others considered paramount: What *is* charge?

3. Manifestly covariant formulation of the preceding material. The emphasis here must be on the "manifestly." The material developed in §2 is relativistic as it stands (as, indeed, were the Maxwell equations (65) on which it is based) . . . but "covertly" so. It will emerge that our recent work becomes much more transparent when rendered in language that makes the Lorentz covariance manifest. We look first to the notational aspects of the matter, then to its transformational aspects (which will be almost obvious):

Let us—in addition to this familiar variant of (159)

$$\|F_{\mu\nu}\| = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

—agree to write

$$\begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} \equiv \begin{pmatrix} \varphi \\ \boldsymbol{A} \end{pmatrix}, \quad \text{equivalently} \quad \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} \varphi \\ -\boldsymbol{A} \end{pmatrix} \tag{368}$$

where the Lorentz metric $g_{\mu\nu}$ has been used to lower the indices. Then equations (363) become

$$\begin{aligned} B_1 = F_{32} &= -F_{23} = -(\partial_2 A_3 - \partial_3 A_2) \\ B_2 = F_{13} &= -F_{31} = -(\partial_3 A_1 - \partial_1 A_3) \\ B_3 = F_{21} &= -F_{12} = -(\partial_1 A_2 - \partial_2 A_1) \\ E_1 = F_{01} &= -F_{10} = -(\partial_1 A_0 - \partial_0 A_1) \\ E_2 = F_{02} &= -F_{20} = -(\partial_2 A_0 - \partial_0 A_2) \\ E_3 = F_{03} &= -F_{30} = -(\partial_3 A_0 - \partial_0 A_3) \end{aligned}$$

or, more compactly,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{369}$$

The preceding construction is obviously invariant under

$$A_\mu \longrightarrow A_\mu = A_\mu + \partial_\mu \chi \tag{370}$$

which when spelled out in detail becomes precisely (364).

The source-independent pair of Maxwell equations were found at (166) to be expressible

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$$

which are seen now to follow *automatically* from the construction (369), while the sourcey pair of Maxwell equations—which at (167) we learned to write

$$\partial_\mu F^{\mu\nu} = \tfrac{1}{c} j^\nu \quad \text{with} \quad \|j^\nu\| \equiv \begin{pmatrix} c\rho \\ \boldsymbol{j} \end{pmatrix}$$

—become $\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \frac{1}{c}j^\nu$ or

$$\square A^\nu - \partial^\nu(\partial_\mu A^\mu) = \tfrac{1}{c}j^\nu \tag{371}$$

The Coulomb gauge condition violates the spirit of relativity (can be adopted by any particular inertial observer, but not simultaneously by all), but that criticism does not pertain to (366), which becomes the

$$\text{LORENTZ GAUGE CONDITION:}\quad \partial_\mu A^\mu = 0 \tag{372}$$

and when in force causes (371) to become

$$\square A^\nu = \tfrac{1}{c}j^\nu \tag{373}$$

which reproduces (367). Imposition of the Lorentz gauge condition does not quite exhaust the available gauge freedom, for

$$\partial_\mu A^\mu = 0 \quad\Longrightarrow\quad \partial_\mu \mathcal{A}^\mu = 0$$
$$\mathcal{A}^\mu = A^\mu + \partial^\mu\chi \quad \text{with } \chi \text{ any solution of } \quad \square\chi = 0$$

It becomes at this point entirely natural to assume that A_μ transforms as a weightless vector field. It is then automatic—here our "catalog of accidentally tensorial derivative constructions" (pages 120–122) comes again into play—that $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ transforms as a weightless antisymmetric tensor, and that $\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$ makes tensorial good sense. On the other hand

- $\partial_\mu F^{\mu\nu} = \frac{1}{c}j^\nu$ is unrestrictedly tensorial if and only if $F^{\mu\nu}$ (whence also j^ν) have *unit* weight
- $\partial_\mu A^\mu = 0$ is unrestrictedly tensorial if and only if A^μ has unit weight

We, however, have interest at the moment in a restricted tensoriality, in *Lorentz covariance* (which means "tensoriality with respect to Lorentz transformations"). Inspection of the arguments used to develop the entries in the "catalog" shows that all weight restrictions arose from the presumption that the elements of the transformation matrix $\mathbb{M} \equiv \|\partial x^m/\partial x^n\|$ change from point to point: $\partial\mathbb{M} \neq \mathbb{O}$. But in that respect the Lorentz transformations—being *linear* transformations —are atypical: one has $\partial\Lambda = \mathbb{O}$, with the consequence that all weight restrictions are lifted. We are brought thus to the conclusion that the numbered equations at the top of the page are *Lorentz covariant as they stand*.

It is now possible—and instructive—to consider afresh this question:

4. So what kind of a thing is Maxwellian electrodynamics? My strategy will be to consider the question not in isolation, but in juxtaposition to a second question: *What kind of a thing is the Proca theory*? . . . and it is to the latter question that we look first.

The Proca theory arises fairly naturally when—within the formal context provided by the "classical theory of fields"—one asks for a **relativistic theory of**

a massive vector field. One is led at length to a system of free-field equations that were encountered already on page 246 and are reproduced below:

$$\partial_\mu U^\mu = 0 \tag{374.1}$$
$$\partial_\mu G^{\mu\nu} + \varkappa^2 U^\nu = 0 \tag{374.2}$$

$$G^{\mu\nu} \equiv \partial^\mu U^\nu - \partial^\nu U^\mu \tag{374.3}$$

$$\partial^\lambda G^{\mu\nu} + \partial^\mu G^{\nu\lambda} + \partial^\nu G^{\lambda\mu} = 0 \tag{374.4}$$

Here U^μ is the physical field, (374.1) and (374.2) are the field equations, (374.3) introduces a notational device used to simplify the statement of the second field equation—which would otherwise read

$$\Box U^\nu \underbrace{-\partial^\nu(\partial_\mu U^\mu)}_{\text{vanishes by the first field equation}} + \varkappa^2 U^\nu = 0$$

—and (374.4) records a corollary property of the "notational device" $G^{\mu\nu}$. Distinct vector fields—namely those that stand in the relationship

$$\color{red}{U^\mu}\color{black} = U^\mu + \partial^\mu\chi$$

—give rise to identical $G^{\mu\nu}$-fields, but the field equations are <u>not invariant</u> under $U^\mu \longrightarrow \color{red}{U^\mu}\color{black} = U^\mu + \partial^\mu\chi$. For if U^μ satisfies

$$\partial_\mu U^\mu = 0$$
$$\partial_\mu G^{\mu\nu} + \varkappa^2 U^\nu = 0$$

then $\color{red}{U^\mu}$ satisfies

$$\partial_\mu(\color{red}{U^\mu}\color{black} - \partial^\mu\chi) = 0$$
$$\partial_\mu G^{\mu\nu} + \varkappa^2(\color{red}{U^\mu}\color{black} - \partial^\mu\chi) = 0$$

which become structurally identical to the original equations if and only if

$$\Box\chi = 0 \quad \text{and} \quad \varkappa^2 = 0$$

In the degenerate case $\varkappa^2 = 0$ the Proca free-field equations (374) become structurally identical to the system of equations that was seen above to describe the free electromagnetic field, but in the latter context the "location of the physics" is shifted, and the equations stand suddenly in a different logical relation to one another. One writes

$$\partial_\mu F^{\mu\nu} = 0 \tag{375.1}$$
$$\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = 0 \tag{375.2}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{375.3}$$

$$\partial_\mu A^\mu = 0 \tag{375.4}$$

What was formerly a mere "notational device" $G^{\mu\nu}$ has now become the *physical* field $F^{\mu\nu}$, and what was formerly dismissed as an incidental "corollary property" has at (375.2) been promoted to the status of a field equation. It is to render that field equation "automatic" that we write (375.3), at which point it is the formerly physical vector field that has acquired the status of a "notational device, a crutch"... denied direct physical significance because it is defined *only up to an arbitrary gauge transformation*. Finally, the Lorentz gauge condition (375.4)—which in Proca theory enjoyed the status of a field equation—has in electrodynamics been demoted to the status of an arbitrarily imposed side condition.

The comparative situation (at least so far as concerns *free* Proca/Maxwell fields: no externally impressed sources/currents) can be summarized this way:

> PROCA has given us the manifestly covariant theory of a **physical/observable massive vector field** U^μ.
>
> MAXWELL has given us (what is in effect, or can be rendered as) the manifestly covariant theory of an **unphysical/unobservable massless vector field**—a "gauge field." The observable physics attaches in that theory to the gauge invariant object
>
> $$\text{field tensor } F^{\mu\nu} \equiv \text{curl of the gauge field}$$

The "theory of gauge fields"—quantum mechanical generalizations of A^μ—has, during the second half of the 20$^{\text{th}}$ Century, moved to center stage in the theory of elementary particles and their fundamental interactions.[211] Our recent experience indicates that *gauge freedom arises from masslessness*, so we are perhaps not surprised to learn that a major problem in that area has been to figure out a way to endow gauge fields with mass (lots of it! ...as the experimental evidence clearly requires). The "Higgs mechanism" stands as the best available solution of the problem,[212] though it is in some respects unattractive, and has as yet no convincing experimental support.

Further insight into the distinctive structure of the electromagnetic field can be gained by carrying "comparative Proca/Maxwell theory" a bit further:

5. Plane wave solutions of the Proca/Maxwell field equations. Both (374) and (375) are notable for their *linearity*. In both theories a *principle of superposition* is operative, so we expect to be able to write

$$\text{general solution} = \sum\!\!\!\!\!\!\int (\text{simple solutions})$$

[211] See L. O'Raifeartaigh, *The Dawning of Gauge Theory* (1997) for a splendid account of the major contours of that development.
[212] See the concluding §11.9 in David Griffiths' *Introduction to Elementary Particles* (1987) for a brief account of the essential idea.

The meaning most usefully assigned to "simple solution" is highly context-dependent (selection of a basis *always* is): it serves my present purpose to proceed as Fourier did; *i.e.*, to write

$$U^\mu(x) = \int \mathsf{U}^\mu(k)\cdot e^{ik\,x}\,d^4k \quad \text{with} \quad \|k^\mu\| \equiv \begin{pmatrix} \omega/c \\ \boldsymbol{k} \end{pmatrix},\ \|x^\mu\| \equiv \begin{pmatrix} ct \\ \boldsymbol{x} \end{pmatrix}$$

where $k\,x \equiv k_\alpha x^\alpha \equiv \omega\,t - \boldsymbol{k}\cdot\boldsymbol{x}$ is evidently Lorentz invariant, where the $\mathsf{U}^\mu(k)$ are understood to transform as a k-parameterized population of complex 4-vectors, and where the reality of $U^\mu(x)$ requires $[\mathsf{U}^\mu(k)]^* = \mathsf{U}^\mu(-k)$. At the expense of some notational clutter we could write

$$U^\mu(x) = \int \mathsf{V}^\mu(k)\cos kx\,d^4k + \int \mathsf{W}^\mu(k)\sin kx\,d^4k$$

where V^μ and W^μ are now understood to be *real* 4-vectors. From the field equations $\big\{g^{\alpha\beta}\partial_\alpha\partial_\beta + \varkappa^2\big\}U^\mu = 0$ and $\partial_\mu U^\mu = 0$ we discover that necessarily

$$k^2 \equiv g^{\alpha\beta}k_\alpha k_\beta \equiv k_0^2 - \boldsymbol{k}\cdot\boldsymbol{k} = \varkappa^2 \tag{376.1}$$

and

$$k_\mu \mathsf{U}^\mu = 0 \quad \text{equivalently} \quad k_\mu \mathsf{V}^\mu = k_\mu \mathsf{W}^\mu = 0 \tag{376.2}$$

The first condition places the k-vector "on the mass shell" (see again Figure 70), while the second condition requires (the real and imaginary parts of) U^μ to be (in the Lorentzian sense) *normal* to k^μ.[213] The question now arises: <u>How many</u> linearly independent vectors V^μ stand normal to any given timelike vector k^μ? The answer, pretty clearly, is <u>three</u>: the following example illustrates the situation

$$\|k^\mu\| = \begin{pmatrix} \varkappa \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \perp \quad \|\mathsf{V}^\mu\| = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ else } \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ else } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and so do all Lorentz transforms of that example.

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[213] The language has become a bit tangled: The mass shell is seen in Figure 70 to live in p-space, while the $\varkappa$-shell lives in k-space. A scale factor distinguishes the one form the other:

$$p = \hbar k \quad \text{and} \quad mc = \hbar\varkappa$$

The "timelike/null (or lightlike)/spacelike" terminology I will carry over from x-space into p-space, though in the latter context it would be more correct to distinguish "energylike" from "momentumlike" 4-vectors. In k-space there is, so far as I am aware, *no* commonly accepted "correct" terminology.

Suppose we write

$$\mathsf{V}^\mu \cos kx = \mathsf{V}^\mu \cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega\, t)$$

to describe one of our "simple free Proca fields." A second inertial observer O would write

$$\mathsf{V}^\mu \cos kx = \mathsf{V}^\mu \cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega\, t)$$

to describe the same physical situation, but we will persist in language special to our own perception of the situation. Writing

$$\phi(\boldsymbol{x}, t) \equiv \boldsymbol{k}\cdot\boldsymbol{x} - \omega\, t \equiv \text{phase}$$

or again

$$\boldsymbol{k}\cdot\boldsymbol{x} = \omega\, t + \text{phase}$$

we see the points of constant phase to lie at time t on a *plane* in 3-dimensional space. From

$$\boldsymbol{\nabla}\phi = \boldsymbol{k} \quad : \quad \text{all } \boldsymbol{x} \text{ and all } t$$

we see that all phase planes stand normal to $\boldsymbol{k}$, which by t-differentiation we have

$$\boldsymbol{k}\cdot\boldsymbol{u} = \omega \quad : \quad \boldsymbol{u} = u\hat{\boldsymbol{k}} \equiv \text{phase velocity}$$

Immediately

$$u = \omega/k = \text{phase speed}$$

From (376.1) we have the "dispersion equation"

$$\omega = c\sqrt{k^2 + \varkappa^2}$$

so

$$u = c\,\frac{\sqrt{k^2+\varkappa^2}}{k} \text{ which } \begin{cases} \text{is } \geqslant c \\ \text{is a } \textit{descending} \text{ function of } k \\ = \infty \text{ at } k = 0 \\ = \ c \ \text{ at } k = \infty \end{cases}$$

On the other hand, we have

$$\begin{aligned} \text{group speed } v &\equiv \frac{d\omega}{dk} \\ &= c\frac{k}{\sqrt{k^2+\varkappa^2}} \text{ which } \begin{cases} \text{is } \leqslant c \\ \text{is an } \textit{ascending} \text{ function of } k \\ = 0 \text{ at } k = 0 \\ = c \text{ at } k = \infty \end{cases} \end{aligned}$$

Different inertial observers will assign different values to u and v, but all will be in agreement that

$$(\text{phase speed}) \cdot (\text{group speed}) = c^2 \quad : \quad \text{all } k$$

The results just developed are standard to all occurrences of the so-called "Klein-Gordon equation" $\Box\psi + \varkappa^2\psi = 0$, which is to say: they are not special to the Proca theory.[214] I turn now to statements that *are* special to the Proca theory. Agree to write $\boldsymbol{e}_\parallel \equiv \hat{\boldsymbol{k}}$, to let $\boldsymbol{e}_1$ be any unit 3-vector normal to $\hat{\boldsymbol{k}}$, and to define $\boldsymbol{e}_2 \equiv \hat{\boldsymbol{k}}\times\boldsymbol{e}_1$, so that $\{\boldsymbol{e}_\parallel, \boldsymbol{e}_1, \boldsymbol{e}_2\}$ comprise a righthanded orthonormal triad in 3-dimensional $\boldsymbol{k}$-space. And recall that $\boldsymbol{k}$ came to us from the 4-vector

$$\|k^\mu\| = \begin{pmatrix} \omega/c \\ \boldsymbol{k} \end{pmatrix} = \begin{pmatrix} \sqrt{k^2+\varkappa^2} \\ \boldsymbol{k} \end{pmatrix} \quad : \quad \text{gives } k_\mu k^\mu = \varkappa^2$$

Now define the spacelike unit 4-vectors

$$\|\mathsf{V}_1^\mu\| \equiv \begin{pmatrix} 0 \\ \boldsymbol{e}_1 \end{pmatrix} \quad \text{and} \quad \|\mathsf{V}_2^\mu\| \equiv \begin{pmatrix} 0 \\ \boldsymbol{e}_2 \end{pmatrix}$$

Clearly

$$\left.\begin{array}{r} \mathsf{V}_1^\mu \perp \mathsf{V}_2^\mu \\ k^\mu \perp \text{ both } \mathsf{V}_1^\mu \text{ and } \mathsf{V}_2^\mu \end{array}\right\} \text{ in the Lorentzian sense}$$

Finally construct

$$\|\mathsf{V}_\parallel^\mu\| \equiv \gamma \begin{pmatrix} \beta \\ \boldsymbol{e}_\parallel \end{pmatrix}$$

NOTE: β and γ are here to be regarded simply as constants, stripped of all prior relativistic associations.

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[214] And though they pertain the the planewave solutions of certain relativistic free fields, the results just obtained bear a striking resemblence to equations encountered in the theory of relativistic free particles . . . for $E = \gamma mc^2$ can be written

$$v = c\,\frac{\sqrt{E^2-(mc^2)^2}}{E}$$

which describes the speed v of a mass m with energy E. We see that

$$\text{particle speed } v \begin{cases} = 0 \text{ at } E = mc^2 \\ \text{approaches } c \text{ as } E\uparrow\infty \\ \therefore \text{ can never equal or exceed } c \end{cases}$$

while in the limit $m\downarrow 0$ we have

$$\text{speed of a "massless particle" is } \textit{always } v = c$$

We see also that the "massless particle" concept is delicate: it would be senseless to write $p^\mu = 0u^\mu$ or $E = \gamma 0c^2$.

and observe that $\perp$ to both V_1^μ and V_2^μ is (for all β) automatic, while $\perp$ k^μ entails $\beta\sqrt{k^2+\varkappa^2}-k=0$, which requires that we set

$$\beta = \frac{k}{\sqrt{k^2+\varkappa^2}} = \frac{\text{group speed } v}{c}$$

The "spacelike unit vector condition"

$$g_{\mu\nu}\mathsf{V}_\parallel^\mu\mathsf{V}_\parallel^\nu = g_{\mu\nu}\mathsf{V}_1^\mu\mathsf{V}_1^\nu = g_{\mu\nu}\mathsf{V}_2^\mu\mathsf{V}_2^\nu = -1$$

requires finally that we set

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

To summarize: the Proca theory supports plane waves of three types. Specification of the propagation vector $\boldsymbol{k}\neq\boldsymbol{0}$ determines both the direction of propagation $\hat{\boldsymbol{k}}$ and the frequency of oscillation $\omega=c\sqrt{k^2+\varkappa^2}$. The three wave types consist of two linearly independent **transverse waves**

$$U^\mu_{\text{transverse}}(x) = \begin{cases} \mathsf{V}_1^\mu\cos(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{x}-\omega\,t+\delta_1) \\ \mathsf{V}_2^\mu\cos(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{x}-\omega\,t+\delta_2) \end{cases}$$

and a solitary **longitudinal wave**

$$U^\mu_{\text{longitudinal}}(x) = \mathsf{V}_\parallel^\mu\cos(\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{x}-\omega\,t+\delta_\parallel)$$

In the degenerate case $\boldsymbol{k}=\boldsymbol{0}$ the "direction of propagation" loses its meaning (there *is* no propagation!), the $\boldsymbol{x}$-dependence drops away, the field oscillates *as a whole* with frequency $\omega_0=c\varkappa$, the "transverse/longitudinal distinction" becomes meaningless, and the orthonormal triad $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\equiv\boldsymbol{e}_\parallel\}$ can be erected arbitrarily. It is as such a "degenerate case" that any Proca field presents itself to any "co-moving observer." The example of page 263 provides an instance of just such a case.

It is in the light of the preceding discussion, and by the formal process $\varkappa^2\downarrow 0$, that we return now to free-field electrodynamics. We have already noted (while discussing the relationship of (375) to (374)) that the transition

$$\varkappa^2 \text{ arbitrarily small} \quad\longrightarrow\quad \varkappa^2=0$$

is formally/qualitatively quite abrupt. The point becomes especially vivid when one looks comparatively to the planewave *solutions* of the Proca/Maxwell field equations. Look first to what happens to the dispersion equation

$$\omega = c\sqrt{k^2+\varkappa^2} \xrightarrow[\varkappa\downarrow 0]{} \omega = ck$$

In Proca theory we found that

$$\text{phase speed} \equiv \omega/k = c\,\frac{\sqrt{k^2+\varkappa^2}}{k} = \frac{\omega}{\sqrt{(\omega/c)^2-\varkappa^2}}$$

is frequency-dependent. Proca fields are "dispersive:" the constituent Fourier components of wavepackets travel at different speeds, and the wavepackets therefore "dissolve." The free electromagnetic field is, on the other hand, *non-dispersive*, since

$$\omega = ck \quad \Longrightarrow \quad \text{phase speed} = \text{group speed} = c \;:\; \text{all } k$$

Look next to what happens to the propagation 4-vector

$$\begin{pmatrix} \sqrt{k^2+\varkappa^2} \\ \boldsymbol{k} \end{pmatrix} \xrightarrow[\varkappa\downarrow 0]{} \begin{pmatrix} k \\ \boldsymbol{k} \end{pmatrix} \quad : \quad \text{clearly a } \textit{null} \text{ vector}$$

By this account, a "co-moving observer"—defined by the condition $\boldsymbol{k} = \mathbf{0}$ —would (because $k^\mu = 0$) see a spatially constant/non-oscillatory potential[215]

$$A^\mu(x) = \mathsf{A}^\mu \quad : \quad \mathsf{A}^\mu \text{ arbitrary}$$
$$\Downarrow$$
$$\text{no electromagnetic } \boldsymbol{E} \text{ or } \boldsymbol{B} \text{ fields at all!}$$

But such use of the "co-moving observer" concept is impossible, for we are informed by Proca theory that such an observer sees the group speed to vanish, while in electrodynamics all inertial observers see the group speed to be c. And it is forbidden to contemplate "inertial observers passing by with the speed of light" because $\mathbb{A}(\boldsymbol{\beta})$ becomes *singular* when $\beta = 1$.

> REMARK: At this point we touch upon a point that engaged the curiosity of the young Einstein, and that contributed later to the invention of special relativity. In his "Autobiographical Notes" (see Paul Schilpp (editor), *Albert Einstein: Philosopher-Scientist* (1951), page 53) he remarks that
>
> > *". . . After ten years of reflection such a principle resulted from a paradox upon which I had already hit at the age of sixteen: if I pursue a light beam with velocity c . . . I should observe such a beam as a spatially oscillatory electromagnetic field at rest. However, there seems to be no such thing, whether on the basis of experience or according to Maxwell's equations . . . It seemed to me intuitively clear that, judged from the standpoint of such an observer, everything would have to happen according to the same laws as for an observer . . . at rest."*
>
> We are in position now to recognize that—beyond his willingness to attach "intuitive clarity" to an impossible fiction —Einstein had (at sixteen!) a somewhat crooked conception of the Maxwellian facts of the matter, but . . .

[215] Forgive the too-casual figure of speech: one cannot "see" electromagnetic 4-potentials, except with the mind's eye!

The transverse Proca plane waves described at the bottom of page 265 go over straightforwardly into **transverse electromagnetic plane waves**: we have

$$A^\mu_{\text{transverse}}(x) = \begin{cases} \mathsf{A}^\mu_1 \cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega\, t + \delta_1) \\ \mathsf{A}^\mu_2 \cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega\, t + \delta_2) \end{cases}$$

where the constant A^μ-vectors differ only notationally from the V^μ-vectors described previously. But Proca's longitudinal plane wave becomes

$$A^\mu_{\text{longitudinal}}(x) = \mathsf{A}^\mu_\parallel \cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega\, t + \delta_\parallel)$$

where in light of the ∞ that intrudes into

$$\|\mathsf{V}^\mu_\parallel\| \equiv \gamma \begin{pmatrix} \beta \\ \boldsymbol{e}_\parallel \end{pmatrix} \xrightarrow[\varkappa\downarrow 0]{} \infty \cdot \begin{pmatrix} 1 \\ \boldsymbol{e}_\parallel \end{pmatrix}$$

we have set

$$\begin{pmatrix} 1 \\ \boldsymbol{e}_\parallel \end{pmatrix} \equiv \mathsf{A}^\mu_\parallel$$

We expected to have $\mathsf{A}^\mu_\parallel \perp k^\mu$, but in fact $\mathsf{A}^\mu_\parallel$ is *parallel* to the propagation vector

$$k^\mu = k \cdot \mathsf{A}^\mu_\parallel$$

and $k_\mu \mathsf{A}^\mu_\parallel = 0$ arises from the circumstance that <u>in electrodynamics k^μ is null</u>. Writing

$$A_\mu^{\text{transverse}}(x) = \frac{\text{constant}}{k} \cdot k_\mu e^{i(k_\alpha x^\alpha)}$$

we find

$$\begin{aligned} F_{\mu\nu}^{\text{transverse}} &= \partial_\mu A_\nu^{\text{transverse}} - \partial_\nu A_\mu^{\text{transverse}} \\ &= i\frac{\text{constant}}{k} \cdot (k_\mu k_\nu - k_\nu k_\mu) \\ &= 0 \end{aligned}$$

and conclude that in electrodynamics the potential $A_\mu^{\text{transverse}}$ can be <u>dismissed as an unphysical artifact</u>:

> Massive Proca fields support three polarizational degrees of freedom, but—"because the photon is massless"—the electromagnetic field supports only two, and they are transverse to the direction of propagation.

As things now stand that statement, by the argument from which it sprang, can be claimed to pertain only to the 4-potential, and to hold only in the Lorentz gauge. But later it will be shown to pertain also to the gauge-independent physical fields $\boldsymbol{E}$ and $\boldsymbol{B}$.

One sometimes encounters attempts to attribute the "disappearance of the longitudinal mode" to the proposition that "an observer riding on a photon sees time dilated to a standstill, and the forward space dimension contracted to extinction." I am not entirely sure the idea actually does what it is intended to do, but in any event: such observers *cannot exist*, so can have no role to play in any convincing account of the physical facts. On the other hand, it is (in other contexts) sometimes illuminating to point out that "an observer riding on a very fast massive particle sees time dilated *nearly* to a standstill, and 3-space contracted *nearly* to a wafer."

6. Contact with the methods of Lagrangian field theory.* As Kermit the Frog might say, "It's not easy, bein' massless"... impossible in pre-relativistic physics, and a delicate business in relativistic physics ... whether you are a particle[214] or a field. Looking to Maxwellian electrodynamics as "Proca theory in the massless limit," we have seen (in §4) that electrodynamics—for all its physical importance—lives *right on the outer edge of formal feasibility*, that "turning off the mass"

- strips the vector field A^μ of its former direct physicality
- introduces "gauge freedom" into the theory
- reduces a formerly basic field equation to the status of a mere convention
- shifts the "locus of physicality," from A^μ to $F^{\mu\nu}$.

In all those respects electromagnetic field is fairly typical of massless fields in general, so close study of the way Maxwell's theory is constructed tends to be more broadly informative than one might at first suppose. The sketchy remarks that follow touch on matters that would be fundamental to any such "close study."

A formalism derived straightforwardly from Lagrangian mechanics is today universally acknowledged to provide the language of choice if one's objective is a systematic development of the properties of a field theory.[216] The formalism in outline: Let φ_a signify the fields of interest.[217] The associated field theory acquires its specific structure from the postulated design of a "Lagrange density" —a real number-valued function $\mathcal{L}(\varphi, \partial\varphi)$ of the field and their spatial/temporal derivatives $\partial_\mu \varphi_a$. An extension of Hamilton's principle

$$\delta S = 0 \quad \text{with} \quad S \equiv \tfrac{1}{c}\int_{\mathcal{R}} \mathcal{L}\, d^4x$$

leads[193] to an a-indexed system of coupled Euler-Lagrange equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \varphi_{a,\,\mu}} - \frac{\partial \mathcal{L}}{\partial \varphi_a} = 0$$

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* This relatively advanced material will *not* be treated in lecture. First-time readers should skip directly to §7.

[216] In some cases of historic importance this was recognized only after the fact: Maxwell, Einstein, Schrödinger, Dirac ... each was led to the field theory that bears his name by methods that made *no* use of the Lagrangian method.

[217] The subscript $_a$ is generic. In specific cases it becomes a set of tensor/spinor indices and other marks used to distinguish one field component from another.

which when spelled out in detail read

$$\frac{\partial^2\mathcal{L}}{\partial\varphi_{a,\,\mu}\partial\varphi_{b,\,\nu}}\varphi_{b,\,\mu\nu} + \frac{\partial^2\mathcal{L}}{\partial\varphi_{a,\,\mu}\partial\varphi_b}\varphi_{b,\,\mu} - \frac{\partial\mathcal{L}}{\partial\varphi_a} = 0$$

These are, in the general case, non-linear partial differential equations into which, however, the second partials enter linearly, and will be manifestly Lorentz covariant if L is Lorentz invariant. Noether's theorem can be brought now into play to deduce the design of the stress-energy tensor and to develop other mechanical properties of the field system, to identify conservation laws, *etc.*

The Proca theory is an unexceptional relativistic field theory that fits straightforwardly into the Lagrangian rubric. Taking the vector field U_μ to be the field system of interest, one constructs[218]

$$\mathcal{L} = \tfrac{1}{2}g^{\alpha\rho}g^{\beta\sigma}U_{\alpha,\beta}(U_{\rho,\sigma} - U_{\sigma,\rho}) - \tfrac{1}{2}\varkappa^2 g^{\alpha\beta}U_\alpha U_\beta$$

and computes

$$\partial_\nu\frac{\partial\mathcal{L}}{\partial U_{\mu,\,\nu}} - \frac{\partial\mathcal{L}}{\partial U_\mu} = \partial_\nu(U^{\mu,\nu} - U^{\nu,\mu}) + \varkappa^2 U^\mu = 0$$

In short: $\Box U^\mu - \partial^\mu(\partial_\nu U^\nu) + \varkappa^2 U^\mu = 0$, which when hit with ∂_μ supplies

$$\begin{aligned} \varkappa^2(\partial_\mu U^\mu) &= 0 \\ &\Downarrow \\ \partial_\mu U^\mu \ &= 0 \quad \text{if} \quad \varkappa^2 \neq 0 \end{aligned}$$

Returning with this information to the field equation, we obtain (see again (374.1&2))

$$\Box U^\mu + \varkappa^2 U^\mu = 0$$

whereupon we might introduce $G_{\mu\nu} \equiv \partial_\mu U_\nu - \partial_\nu U_\mu$ as an auxiliary definition. *Alternatively*, we might take $\big\{U_\mu, G_{\mu\nu}\big\}$ to be the field system of interest, and write

$$\mathcal{L} = -\tfrac{1}{4}g^{\alpha\rho}g^{\beta\sigma}G_{\alpha\beta}G_{\rho\sigma} - \tfrac{1}{2}g^{\alpha\rho}g^{\beta\sigma}G_{\alpha\beta}(U_{\rho,\sigma} - U_{\sigma,\rho}) - \tfrac{1}{2}\varkappa^2 g^{\alpha\beta}U_\alpha U_\beta$$

giving

$$\begin{aligned} \partial_\kappa\frac{\partial\mathcal{L}}{\partial G_{\mu\nu,\,\kappa}} - \frac{\partial\mathcal{L}}{\partial G_{\mu\nu}} &= \quad\ \tfrac{1}{2}G^{\mu\nu} + \tfrac{1}{2}(U^{\mu,\nu} - U^{\nu,\mu}) = 0 \\ \partial_\nu\frac{\partial\mathcal{L}}{\partial U_{\mu,\,\nu}} - \frac{\partial\mathcal{L}}{\partial U_\mu} &= -\tfrac{1}{2}\partial_\nu(G^{\mu\nu} - G^{\nu\mu}) + \varkappa^2 U^\mu = 0 \end{aligned}$$

[218] See my CLASSICAL FIELD THEORY (1999), Chapter 2, pages 16–19 for discussion of why this is a relativistically natural thing to do, and for other details.

The former "auxiliary definition" has now acquired the status of a field equation

$$G_{\mu\nu} = \partial_\mu U_\nu - \partial_\nu U_\mu$$

The automatic antisymmetry of $G_{\mu\nu}$ permits the second set of field equations to be written

$$\partial_\nu G^{\nu\mu} + \varkappa^2 U^\mu = 0$$

and from that pair of equations we again recover $\partial_\mu U^\mu = 0$ as a corollary *provided* $\varkappa^2 \neq 0$.

Proca theory supplies us with a way to construct $G_{\mu\nu}$ from U_μ but no way to construct U_μ from $G_{\mu\nu}$. It is therefore not possible to dismiss U_μ from the list of field functions, to consider Lagrangians of the form $\mathcal{L}(G, \partial G)$. Nor are we motivated to do so. But in electrodynamics—where $F_{\mu\nu}$ is physical but the vector field A_μ is *un*physical—that would be our natural instinct. It appears, however, to be impossible to obtain the free-field Maxwell equations

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= 0 \\ \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} &= 0 \end{aligned}$$

from a Lagrangian of the form $\mathcal{L}(F, \partial F)$: we are *forced to enlist the assistance of the 4-potential*. . . and then things become easy. If, for example, we borrow from Proca theory the construction[219]

$$\mathcal{L} = -\tfrac{1}{4} g^{\alpha\rho} g^{\beta\sigma} F_{\alpha\beta} F_{\rho\sigma} - \tfrac{1}{2} g^{\alpha\rho} g^{\beta\sigma} F_{\alpha\beta}(A_{\rho,\sigma} - A_{\sigma,\rho}) + \text{no mass term}$$

then we obtain

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{whence} \quad \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$$

and

$$\partial_\mu F^{\mu\nu} = 0$$

but because $\varkappa^2 = 0$ we have lost the leverage which would enforce the Lorentz gauge condition $\partial_\mu A^\mu = 0$.

The preceding discussion touches on yet another sense in which Maxwellian electrodyanmics is—for the familiar reason ("masslessness of the photon")—formally exceptional, delicate.

7. Naked potential in the classical/quantum dynamics of particles. Though particles respond to *forces* $\boldsymbol{F} = -\boldsymbol{\nabla} U$, it is the *naked potential* that enters into the design of the Lagrangian $L = T - U$ (which, as was remarked on page 254, is itself a kind of "potential"). We found at (293) that the non-relativistic[220]

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[219] For other possibilities see A. O. Barut, *Electrodynamics and Classical Theory of Fields and Particles* (1964), page 102.

[220] Why *non*-relativistic? Because my destination is a result that emerges from non-relativistic quantum mechanics.

motion of a charged particle in an impressed electromagnetic field can be described

$$\tfrac{d}{dt}(m\boldsymbol{v}) = e\big\{\boldsymbol{E} + \tfrac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\big\}$$

which by (363) becomes

$$= e\big\{-\boldsymbol{\nabla}\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A} + \tfrac{1}{c}\boldsymbol{v}\times\boldsymbol{\nabla}\times\boldsymbol{A}\big\} \tag{377}$$

I begin this discussion with a review of how that equation of motion can be brought within the compass of Lagrangian mechanics. We will not be surprised when we find that φ and $\boldsymbol{A}$ stand nakedly/undifferentiated in our final result.

From $\frac{d}{dt}\boldsymbol{A} = \frac{\partial}{\partial t}\boldsymbol{A} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A}$ it follows that

$$-\tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A} = -\tfrac{1}{c}\tfrac{d}{dt}\boldsymbol{A} + \tfrac{1}{c}(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A}$$

Moreover

$$\tfrac{1}{c}\boldsymbol{v}\times\boldsymbol{\nabla}\times\boldsymbol{A} = \tfrac{1}{c}\boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{A}) - \tfrac{1}{c}(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{A}$$

Taken in combination, those two identities supply

$$e\big\{-\boldsymbol{\nabla}\varphi - \tfrac{1}{c}\tfrac{\partial}{\partial t}\boldsymbol{A} + \tfrac{1}{c}\boldsymbol{v}\times\boldsymbol{\nabla}\times\boldsymbol{A}\big\} = e\big\{-\boldsymbol{\nabla}\varphi - \tfrac{1}{c}\tfrac{d}{dt}\boldsymbol{A} + \tfrac{1}{c}\boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{A})\big\}$$

But

$$\begin{aligned} e\big\{-\boldsymbol{\nabla}\varphi - \tfrac{1}{c}\tfrac{d}{dt}\boldsymbol{A} + \tfrac{1}{c}\boldsymbol{\nabla}(\boldsymbol{v}\cdot\boldsymbol{A})\big\}_i &= e\big\{-\varphi_{,i} - \tfrac{1}{c}\tfrac{d}{dt}A_i + \tfrac{1}{c}\boldsymbol{v}\cdot\boldsymbol{A}_{,i}\big\} \\ &= \left\{\frac{d}{dt}\frac{\partial}{\partial v_i} - \frac{\partial}{\partial x_i}\right\}e(\varphi - \tfrac{1}{c}\boldsymbol{v}\cdot\boldsymbol{A}) \end{aligned}$$

The implication is that (377) can be written

$$\begin{aligned} &\left\{\frac{d}{dt}\frac{\partial}{\partial v_i} - \frac{\partial}{\partial x_i}\right\}L = 0 \\ &L \equiv \tfrac{1}{2}m\boldsymbol{v}\cdot\boldsymbol{v} - \underbrace{e(\varphi - \tfrac{1}{c}\boldsymbol{v}\cdot\boldsymbol{A})} \end{aligned} \tag{378}$$

Classic instance of a "velocity-dependent potential" that gives rise by Lagrange differentiation to a velocity-dependent force: see, for example, Goldstein's Section I-5.

As anticipated, the potentials stand naked in L.

The "momentum conjugate to $\boldsymbol{x}$" is given by

$$\boldsymbol{p} \equiv \frac{\partial L}{\partial \boldsymbol{v}} = m\boldsymbol{v} + \tfrac{e}{c}\boldsymbol{A} \tag{379}$$

and must be distinguished from the "mechanical momentum" $m\boldsymbol{v}$. Substitution of $\boldsymbol{v} = \frac{1}{m}\big(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\big)$ into $H = \boldsymbol{v}\cdot\boldsymbol{p} - L(\boldsymbol{x},\boldsymbol{v})$ gives the associated Hamiltonian

$$H(\boldsymbol{x},\boldsymbol{p}) = \tfrac{1}{2m}\big(\boldsymbol{p} - \tfrac{e}{c}\boldsymbol{A}\big)\cdot\big(\boldsymbol{p} - \tfrac{e}{c}\boldsymbol{A}\big) + e\varphi \tag{380}$$

Though the motion must necessarily be gauge invariant, the Lagrangian is not: the gauge transformation (364)

$$\begin{aligned}\varphi &\longrightarrow \varphi = \varphi + \tfrac{1}{c}\tfrac{\partial}{\partial t}\chi\\ \boldsymbol{A} &\longrightarrow \boldsymbol{A} = \boldsymbol{A} - \boldsymbol{\nabla}\chi\end{aligned}$$

sends

$$\begin{aligned}L \longrightarrow L &= \tfrac{1}{2}m\boldsymbol{v}\cdot\boldsymbol{v} - e(\varphi - \tfrac{1}{c}\boldsymbol{v}\cdot\boldsymbol{A})\\ &= L - \tfrac{e}{c}\big\{\tfrac{\partial}{\partial t}\chi + \boldsymbol{v}\cdot\boldsymbol{\nabla}\chi\big\}\\ &= L - \tfrac{d}{dt}\big\{\tfrac{e}{c}\chi\big\}\end{aligned}$$

From the final equation we conclude that, though L and L are distinct, they are (see again page 254) *gauge-equivalent in the sense of Lagrangian mechanics*—in the sense, that is to say, that they give rise to identical Lagrange equations. The *action* associated with any Hamiltonian test-path $\boldsymbol{x}(t)$

$$S[\boldsymbol{x}(t)] \equiv \int_{t_1}^{t_2} L\big(\boldsymbol{x}(t), \boldsymbol{v}(t)\big)\,dt$$

therefore responds to gauge transformation by a rule

$$S \longrightarrow S = S - \tfrac{e}{c}\big\{\chi(\boldsymbol{x}_2) - \chi(\boldsymbol{x}_1)\big\} \tag{381}$$

in which for the first time we see the "naked gauge function" (evaluated here at the specified endpoints of the test-path: $\boldsymbol{x}_1 \equiv \boldsymbol{x}(t_1)$ and $\boldsymbol{x}_2 \equiv \boldsymbol{x}(t_2)$).

Turning now from the classical to the *quantum* mechanics of a charged particle in an impressed field, we are led from (380) to the time-dependent Schrödinger equation

$$\begin{aligned}\mathsf{H}\psi = i\hbar\tfrac{\partial}{\partial t}\psi \quad\text{with}\quad \mathsf{H} &\equiv \tfrac{1}{2m}\big(\tfrac{\hbar}{i}\boldsymbol{\nabla} - \tfrac{e}{c}\boldsymbol{A}\big)\cdot\big(\tfrac{\hbar}{i}\boldsymbol{\nabla} - \tfrac{e}{c}\boldsymbol{A}\big) + e\varphi\\ &= -\tfrac{\hbar^2}{2m}\big(\boldsymbol{\nabla} - ig\boldsymbol{A}\big)\cdot\big(\boldsymbol{\nabla} - ig\boldsymbol{A}\big) + e\varphi\\ g &\equiv e/\hbar c\end{aligned} \tag{382}$$

We expect/require the quantum physics to be gauge-invariant, but observe that H is clearly *not* gauge-invariant. As a first step toward reconciling the latter fact with the former requirement we observe (i) that the Schrödinger equation can be written

$$-\tfrac{\hbar^2}{2m}\big(\boldsymbol{\nabla} - ig\boldsymbol{A}\big)\cdot\big(\boldsymbol{\nabla} - ig\boldsymbol{A}\big)\psi = i\hbar(\tfrac{\partial}{\partial t} + igc\varphi)\psi$$

and (ii) that from the "shift rule"

$$e^{-F(u)}\tfrac{\partial}{\partial u}\bullet \equiv \big[\tfrac{\partial}{\partial u} + \tfrac{\partial F}{\partial u}\big]e^{-F(u)}\bullet$$

it follows that if we multiply the left/right sides of the Schrödinger equation by $e^{-ig\chi}$ we obtain an equation that can be written

$$-\tfrac{\hbar^2}{2m}\big(\boldsymbol{\nabla}+ig\boldsymbol{\nabla}\chi-ig\boldsymbol{A}\big)\boldsymbol{\cdot}\big(\boldsymbol{\nabla}+ig\boldsymbol{\nabla}\chi-ig\boldsymbol{A}\big)e^{-ig\chi}\psi = i\hbar\big(\tfrac{\partial}{\partial t}+ig\tfrac{\partial\chi}{\partial t}+igc\varphi\big)e^{-ig\chi}\psi$$

or again

$$-\tfrac{\hbar^2}{2m}\big(\boldsymbol{\nabla}-ig{\color{red}\boldsymbol{A}}\big)\boldsymbol{\cdot}\big(\boldsymbol{\nabla}-ig{\color{red}\boldsymbol{A}}\big)e^{-ig\chi}\psi = i\hbar\big(\tfrac{\partial}{\partial t}+igc{\color{red}\varphi}\big)e^{-ig\chi}\psi$$

The implication is that if we interpret "gauge transformation" to have this expanded meaning

$$\left.\begin{aligned} \varphi &\longrightarrow {\color{red}\varphi} = \varphi + \tfrac{1}{c}\tfrac{\partial}{\partial t}\chi \\ \boldsymbol{A} &\longrightarrow {\color{red}\boldsymbol{A}} = \boldsymbol{A} - \boldsymbol{\nabla}\chi \\ \psi &\longrightarrow {\color{red}\psi} = e^{-ig\chi}\cdot\psi \end{aligned}\right\} \tag{383}$$

then we have achieved a *gauge-covariant quantum theory*

$$\begin{aligned} &\Big\{-\tfrac{\hbar^2}{2m}\big(\boldsymbol{\nabla}-ig\boldsymbol{A}\big)\boldsymbol{\cdot}\big(\boldsymbol{\nabla}-ig\boldsymbol{A}\big)+e\varphi\Big\}\psi = i\hbar\tfrac{\partial}{\partial t}\psi \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Big\downarrow \text{gauge transformation} \\ &\Big\{-\tfrac{\hbar^2}{2m}\big(\boldsymbol{\nabla}-ig{\color{red}\boldsymbol{A}}\big)\boldsymbol{\cdot}\big(\boldsymbol{\nabla}-ig{\color{red}\boldsymbol{A}}\big)+e{\color{red}\varphi}\Big\}{\color{red}\psi} = i\hbar\tfrac{\partial}{\partial t}{\color{red}\psi} \end{aligned}$$

which—more to the point—yields gauge-invariant physical statements, of which the following

$$\begin{aligned} (\psi|\mathbf{x}|\psi) &= ({\color{red}\psi}|{\color{red}\mathbf{x}}|{\color{red}\psi}) \\ (\psi|\mathbf{p}-\tfrac{e}{c}\mathbf{A}|\psi) &= ({\color{red}\psi}|{\color{red}\mathbf{p}}-\tfrac{e}{c}{\color{red}\mathbf{A}}|{\color{red}\psi}) \end{aligned}$$

are merely illustrative.

To retain the relative simplicity of time-independent quantum mechanics, let us assume for the moment that all potentials and gauge functions depend only upon $\boldsymbol{x}$. We are placed then in position to write

$$\psi(\boldsymbol{x},t) = \int G(\boldsymbol{x},t;\boldsymbol{x}_0,0)\,\psi(\boldsymbol{x}_0,0)\,d^3x_0$$

and thus to describe the *temporal evolution* of the (unobserved) wavefunction. Quantum mechanics provides two alternative descriptions of the "propagator" $G(\boldsymbol{x},t;\boldsymbol{x}_0,t_0)$: the "spectral description"

$$G(\boldsymbol{x},t;\boldsymbol{x}_0,0) = \sum_n e^{-\frac{i}{\hbar}E_n t}\psi_n(\boldsymbol{x})\psi_n^*(\boldsymbol{x}_0)$$

and Feynman's "sum-over-paths description"

$$G(\boldsymbol{x},t;\boldsymbol{x}_0,0) = (\text{normalization factor})\cdot\sum_{\text{paths}}\exp\Big\{\tfrac{i}{\hbar}S[\text{path: }(\boldsymbol{x}_0,0)\rightarrow(\boldsymbol{x},t)]\Big\}$$

Bringing $\psi = e^{ig\chi}\cdot{\color{red}\psi}$ to the spectral description we obtain

$$G(\boldsymbol{x},t;\boldsymbol{x}_0,0) = {\color{red}G}(\boldsymbol{x},t;\boldsymbol{x}_0,0)\cdot\exp\big\{ig\big[\chi(\boldsymbol{x})-\chi(\boldsymbol{x}_0)\big]\big\}$$

The point of interest is that since (381) can be expressed

$$\tfrac{i}{\hbar}S[\text{path}] = \tfrac{i}{\hbar}{\color{red}S}[\text{path}] + ig\big[\chi(\boldsymbol{x})-\chi(\boldsymbol{x}_0)\big]$$

the Feynman method leads immediately to that same conclusion,[221] and does so independently of how we elect to give meaning to the "sum-over-paths" concept.

From $\boldsymbol{B} = \boldsymbol{\nabla}\times \boldsymbol{A}$ it follows that

$$\begin{aligned}\text{magnetic flux } \varPhi \text{ through disk bounded by } \mathcal{C} &= \iint_{\text{disk}} \boldsymbol{B}\cdot \boldsymbol{d\sigma}\\ &= \iint_{\text{disk}} (\boldsymbol{\nabla}\times \boldsymbol{A})\cdot \boldsymbol{d\sigma}\\ &= \oint_{\mathcal{C}} \boldsymbol{A}\cdot \boldsymbol{d\ell} \end{aligned} \tag{384}$$

This simple result is of importance for at least *two* reasons:

1. It exposes a *gauge-independent* "naked $\boldsymbol{A}$":

$$\oint_{\mathcal{C}} \boldsymbol{A}\cdot \boldsymbol{d\ell} = \oint_{\mathcal{C}} \boldsymbol{A}\cdot \boldsymbol{d\ell} \quad \text{because} \quad \oint_{\mathcal{C}} \boldsymbol{\nabla}\chi\cdot \boldsymbol{d\ell} = 0 \ (\text{all } \chi)$$

2. It assigns physical importance (as explained below) to certain topological circumstances, and does so for reasons that are of some interest in themselves. The simplest way to expose the points at issue is to consider the "cylindrical" magnetic field shown in Figure 84. The symmetry of the field, and what we know about the geometrical meaning of "curl," suggest that the $\boldsymbol{A}$-field should have (to within gauge) the form indicated in Figure 85:

$$\boldsymbol{A} = A(r)\,\boldsymbol{T}$$

$$\boldsymbol{T} \equiv \text{unit tangent to Amperian circle of radius } r = \begin{pmatrix} -y/r \\ +x/r \\ 0 \end{pmatrix}$$

Working from (384) we therefore have

$$\text{encircled flux} = \begin{cases} \pi\, r^2 B \text{ if } r \leqslant R \\ \pi R^2 B \text{ if } r \geqslant R \end{cases} = 2\pi r \cdot A(r)$$

[221] Or would if we could establish the gauge-independence of the normalization factor. The point becomes trivial if one is willing to borrow from the result of the spectral argument, but (except in the simplest cases) is too intricate to pursue here by methods internal to the Feynman formalism. Evaluation of the normalization factor is in some respects the most delicately problematic aspect of the formalism. Feynman himself was content to assume that

$$\text{normalization factor} = (\boldsymbol{x}, \boldsymbol{x}_0)\text{-independent function of } t$$

and to extract its specific design from the requirement that

$$\lim_{t\downarrow 0} G(\boldsymbol{x}, t; \boldsymbol{x}_0, 0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0)$$

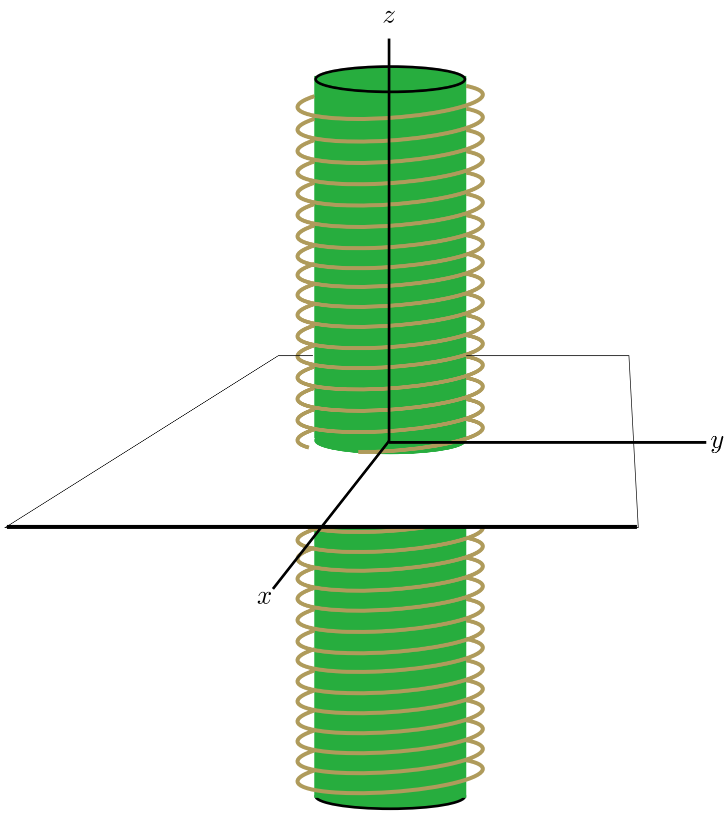

FIGURE 84: *Current flows in an infinitely long straight solenoid, of radius R. The resulting magnetic field is well known to be coaxial and uniform across the interior of the solenoid, but to vanish at all points exterior to the solenoid:*

$$\boldsymbol{B} = \begin{cases} \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} & \textit{at interior points} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & \textit{at exterior points} \end{cases}$$

The "magnetic spaghetti, stretching from one side of Euclidean space to the other," alters the topology of the part of space where $\boldsymbol{B} = \mathbf{0}$*, and this is shown in the text to have some profound physical consequences. Additional spaghetti would make the topological situation even more complicated. The configuration shown has the merit of being simple enough to permit all calculations to be done exactly.*

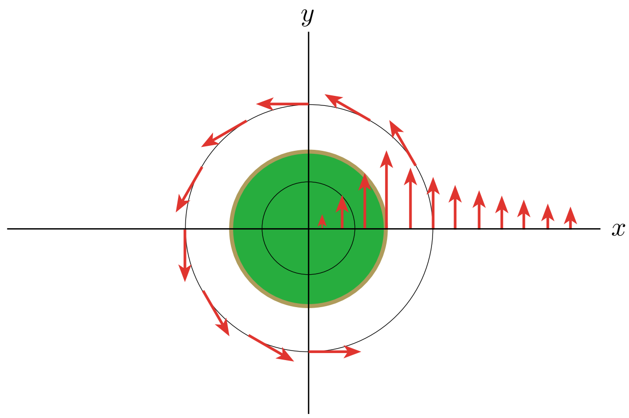

FIGURE 85: *Cross-section of the preceding figure. The black circles (one with radius $r < R$, one with $r > R$) are ↺-oriented "Amperian loops" drawn to capture the symmetry of the system. Red arrows decorate the larger loop, and indicate the anticipated design of the* $\boldsymbol{A}$*-field. The red arrows that march along the x-axis illustrate how the magnitude of* $\boldsymbol{A}$*, as computed in the text, depends upon r. The striking fact is that,* <u>*while* $\boldsymbol{B}$ *vanishes at exterior points $r > R$, the vector potential* $\boldsymbol{A}$ *does not*</u>.

A little guesswork has brought us thus to

$$A(r) = \begin{cases} \tfrac{1}{2}Br & : \quad r \leqslant R \\ \tfrac{1}{2}BR^2 r^{-1} & : \quad r \geqslant R \end{cases}$$

whence

$$\boldsymbol{A}(\boldsymbol{x}) = \begin{cases} \tfrac{1}{2}B\begin{pmatrix} -y \\ +x \\ 0 \end{pmatrix} & : \quad r \leqslant R \\ \\ \tfrac{1}{2}BR^2\begin{pmatrix} -y/r^2 \\ +x/r^2 \\ 0 \end{pmatrix} & : \quad r \geqslant R \end{cases} \tag{385}$$

and a quick calculation[222] confirms the accuracy of the guess:

$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} = \begin{cases} \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} & : \quad r \leqslant R \\ \boldsymbol{0} & : \quad r \geqslant R \end{cases}$$

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[222] PROBLEM 59.

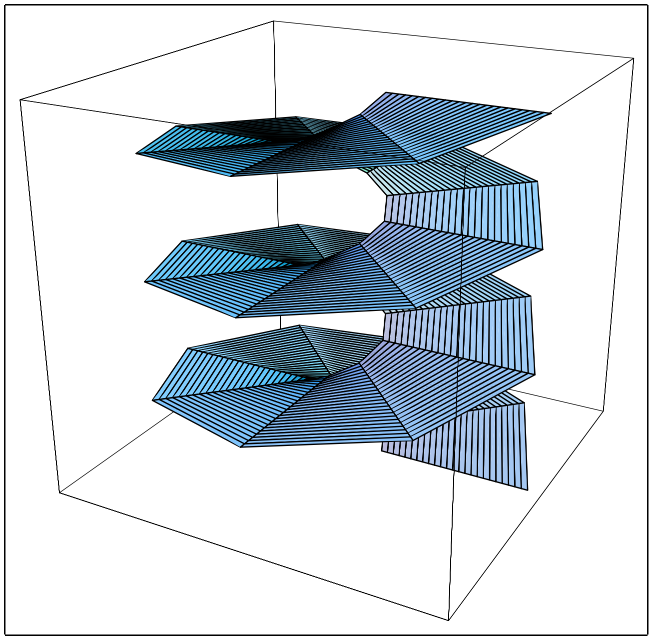

FIGURE 86: *Graph of the multivalued superpotential* $\alpha(x, y)$ *defined at (387)*

In the exterior region the condition $\boldsymbol{\nabla}\times\boldsymbol{A} = \boldsymbol{0}$ would be rendered automatic if we wrote

$$\boldsymbol{A} = \boldsymbol{\nabla}\alpha \tag{386}$$
$$\uparrow\text{“superpotential”}$$

The $\boldsymbol{A}$-vectors stand normal to the equi-(super)potential surfaces, so from results in hand we infer that $\alpha(\boldsymbol{x})$ is constant on planes that radiate radially from the z-axis: $\alpha(\boldsymbol{x}) = f(\arctan(y/x))$. On a hunch, we try the simplest instance of such a function

$$\alpha(x, y, z) = \tfrac{1}{2}BR^2 \arctan(y/x) \tag{387}$$

and by quick calculation (ask *Mathematica*) verify that indeed

$$\boldsymbol{\nabla}\alpha = \tfrac{1}{2}BR^2 \begin{pmatrix} -y/r^2 \\ +x/r^2 \\ 0 \end{pmatrix} = \boldsymbol{A}_{\text{exterior}}$$

The superpotential defined at (387) is plotted in Figure 86. It is clearly multivalued, but—a remark of David Griffiths[223] notwithstanding—no physical principle excludes that possibility: we are concerned here not with potentials but with *super*potentials.

[223] *Introduction to Electrodynamics* (1981), page 207, Problem 29.

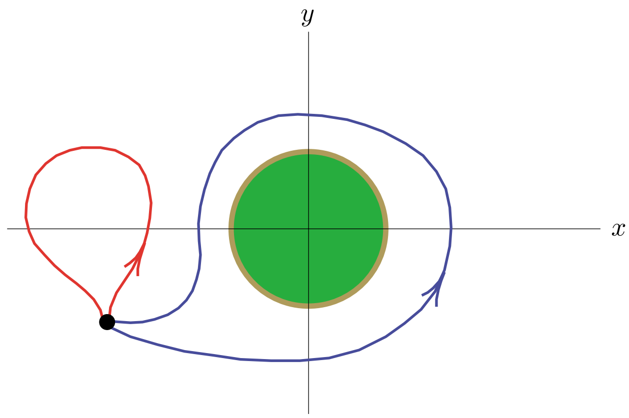

FIGURE 87: *Two (oriented) curves are inscribed on a plane from which a single green hole has been excised. Each curve begins & ends at the point marked •. Many curves are equivalent to—the proper phrase is "homotopic to"—the red curve $\mathcal{C}$ in the sense that they could be brought into coincidence with $\mathcal{C}$ by continuous deformation. But the blue curve $\mathcal{C}$ is not among them: the required deformation is impeded by the circumstance that $\mathcal{C}$ winds (once) around the hole. Evidently $\mathcal{C}_1$ and $\mathcal{C}_2$ will be homotopically equivalent*

$$\mathcal{C}_1 \sim \mathcal{C}_2 \text{ iff } \mathcal{C}_1 \text{ and } \mathcal{C}_2 \text{ have the same "winding number"}$$

The idea of resolving the set of all •-based curves into homotopic equivalence classes extends straightforwardly to more complex situations (multiple holes in the plane, surfaces of sphere/torus/etc.). Down this road lies "homotopy theory," of which a very good introductory account (written for physicists) can be found in §23.2 of L. S. Schulman's Techniques & Applications of Path Integration (1981).

I allude above to the topological information that can be gained from resolving curves/loops/paths into homotopic equivalence classes. Some physical problems hinge naturally on precisely that mode of classification, and acquire thus a "topological" aspect. One such—but by no means the only such—problem was identified by Bohm & Aharonov in 1959,[224] who contemplate a modification of the "two slit experiment" in which (see Figure 88) a solenoid is tucked behind the slits: particles, in their flight from source to detector, experience no electromagnetic forces, but pass through a region in which $\boldsymbol{A} \neq \boldsymbol{0}$, and the latter circumstance has (as Bohm & Aharonov were actually not the first to point out) observable consequences. I turn now to a sketch of how the so-called "Bohm-Aharonov effect" comes about:

[224] Y. Aharonov & D. Bohm, "Significance of electromagnetic potentials in the quantum theory," Phys. Rev. **115**, 485 (1959).

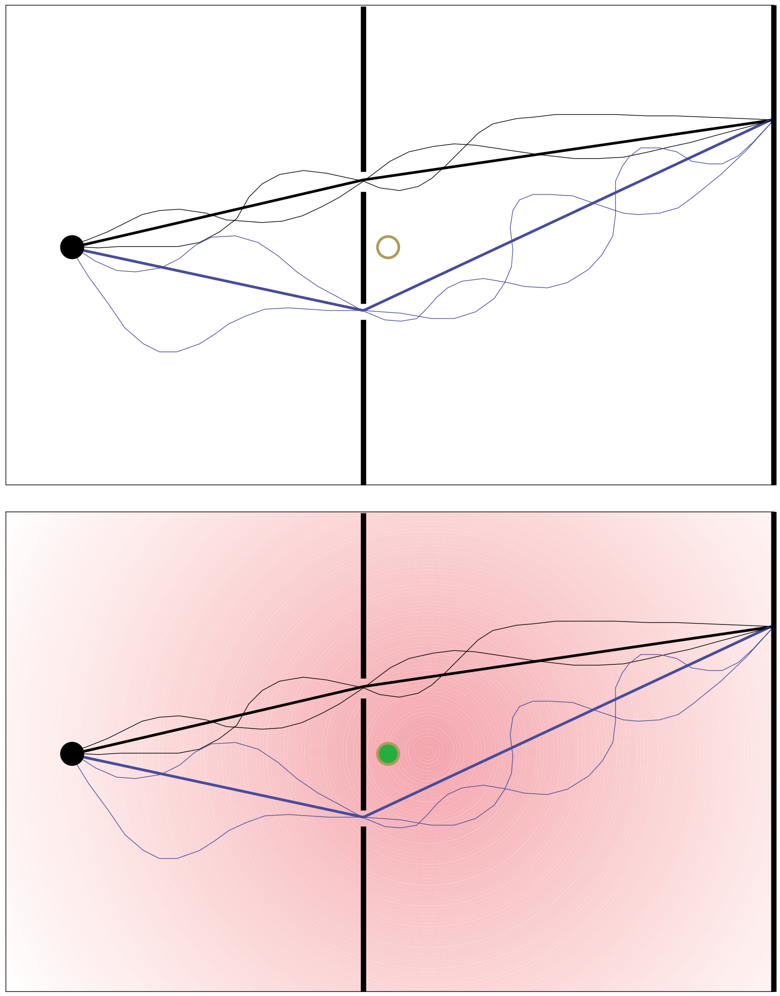

FIGURE 88: *In Bohm/Aharonov's modification (below) of the classic 2-slit experiment (above) a solenoid produces a localized* $\boldsymbol{B}$*-field. By arrangement, particles—in their flight from source to detector—are excluded from the region where* $\boldsymbol{B} \neq \mathbf{0}$*, but pass through a region now flooded with the associated* $\boldsymbol{A}$ *field. The latter circumstance was predicted and experimentally found to cause an observable alteration of the interference pattern—the* **Bohm-Aharonov effect**.

In the classic 2-slit set-up (prior to Bohm/Aharonov's modification) a particle proceeds in time t from source *via* slit #1 to detection point $\boldsymbol{x}$ with probability amplitude

$$\psi_1(\boldsymbol{x},t) \sim \sum_{\text{such paths}} e^{\frac{i}{\hbar}S[\text{path via slit \#1}]} \tag{388}$$

where the $\sim$ signals my intention to be casual about normalization factors throughout this discussion. $\psi_2(\boldsymbol{x},t)$ is defined similarly, and the *net* amplitude for arrival at $(\boldsymbol{x},t)$ is given by

$$\psi(\boldsymbol{x},t) = \psi_1(\boldsymbol{x},t) + \psi_2(\boldsymbol{x},t)$$

All three of those functions are solutions of

$$-\tfrac{\hbar^2}{2m}\nabla^2\psi = i\hbar\tfrac{\partial}{\partial t}\psi$$

though ψ_1 and ψ_2 satisfy somewhat different boundary conditions (ψ_1 vanishes at slit #2, ψ_2 vanishes at slit #1). If we write

$$\psi_1(\boldsymbol{x},t) = \sqrt{P_1(\boldsymbol{x},t)}\,e^{i\phi_1(\boldsymbol{x},t)} \quad\text{and}\quad \psi_2(\boldsymbol{x},t) = \sqrt{P_2(\boldsymbol{x},t)}\,e^{i\phi_2(\boldsymbol{x},t)}$$

then the *probability* of detection at $(\boldsymbol{x},t)$ is given by

$$\begin{aligned} P(\boldsymbol{x},t) &= |\psi_1+\psi_2|^2 \\ &= P_1 + P_2 + \underbrace{2\sqrt{P_1P_2}\cos\Delta\phi}_{\text{interference term}} \end{aligned}$$

Here $\Delta\phi \equiv \phi_1 - \phi_2$ and we dismiss as irrelevant the fact that most detectors are so slow that they report only the value of $P(\boldsymbol{x}) \equiv \int_0^\infty P(\boldsymbol{x},t)\,dt$.

Now turn on the current in the solinoid. In place of (388) we have

$$\color{red}{\psi}_1(\boldsymbol{x},t) \sim \sum_{\text{paths}} e^{\frac{i}{\hbar}\left\{S[\text{path via slit \#1}]+\frac{e}{c}\int_{\text{path}}\boldsymbol{A}\cdot\boldsymbol{dx}\right\}}$$

But all paths $\bullet \xrightarrow[\text{via slit \#1}]{} \boldsymbol{x}$ (let such paths be called "paths of type #1") are homotopically equivalent, $\boldsymbol{\nabla}\times\boldsymbol{A} = \boldsymbol{0}$ holds at every point along each, so we have

$$\begin{aligned} \int_{\text{any path of type \#1}}\boldsymbol{A}\cdot\boldsymbol{dx} &= \int_{\text{any other such path}}\boldsymbol{A}\cdot\boldsymbol{dx} \\ &= \text{path-independent function of } \boldsymbol{x} \end{aligned}$$

giving

$$\color{red}{\psi}_1(\boldsymbol{x},t) = e^{\frac{i}{\hbar}\frac{e}{c}\int_{\text{typical path of type \#1}}\boldsymbol{A}\cdot\boldsymbol{dx}}\cdot\psi_1(\boldsymbol{x},t)$$

We note in passing that from the operator identity

$$\boldsymbol{\nabla} = e^{-\frac{i}{\hbar}\frac{e}{c}\int_{\#1}\boldsymbol{A}\cdot\boldsymbol{dx}}\left[\boldsymbol{\nabla} - \tfrac{i}{\hbar}\tfrac{e}{c}\boldsymbol{A}\right]e^{\frac{i}{\hbar}\frac{e}{c}\int_{\#1}\boldsymbol{A}\cdot\boldsymbol{dx}}$$

it follows that if ψ_1 satisfies the Schrödinger equation at the top of the page then $\color{red}{\psi}_1$ satisfies

$$-\tfrac{\hbar^2}{2m}\left[\boldsymbol{\nabla} - ig\boldsymbol{A}\right]\cdot\left[\boldsymbol{\nabla} - ig\boldsymbol{A}\right]\color{red}{\psi}_1 = i\hbar\tfrac{\partial}{\partial t}\color{red}{\psi}_1$$

—as expected. Identical remarks pertain, of course, to $\color{red}{\psi}_2$.

Which brings us at last to the main point of this discussion. It follows from results now in hand that turning on the solenoidal $\boldsymbol{B}$-field sends

$$P(\boldsymbol{x}) = P_1(\boldsymbol{x}) + P_2(\boldsymbol{x}) + 2\sqrt{P_1(\boldsymbol{x})P_2(\boldsymbol{x})}\cos\Big\{\Delta\phi(\boldsymbol{x})\Big\}$$
$$\downarrow$$
$$P(\boldsymbol{x}) = P_1(\boldsymbol{x}) + P_2(\boldsymbol{x}) + 2\sqrt{P_1(\boldsymbol{x})P_2(\boldsymbol{x})}\cos\Big\{\Delta\phi(\boldsymbol{x}) + g\Big[\int_{\#1} - \int_{\#2}\Big]\boldsymbol{A}\cdot d\boldsymbol{x}\Big\}$$

But paths $\bullet \xrightarrow[\text{via slit }\#1]{} \boldsymbol{x}$ and $\bullet \xrightarrow[\text{via slit }\#2]{} \boldsymbol{x}$ are homotopically inequivalent: the integrals, instead of cancelling, produce

$$\oint_{\mathcal{C}} \boldsymbol{A}\cdot d\boldsymbol{x} = \text{enveloped magnetic flux } \varPhi$$

because $\mathcal{C}$ encloses the solenoid. So we have

$$P(\boldsymbol{x}) = P_1(\boldsymbol{x}) + P_2(\boldsymbol{x}) + 2\sqrt{P_1(\boldsymbol{x})P_2(\boldsymbol{x})}\cos\Big\{\Delta\phi(\boldsymbol{x}) + \tfrac{e}{\hbar c}\varPhi\Big\} \tag{389}$$

which, since $\varPhi$ is $\boldsymbol{x}$-independent, describes an *observably shifted copy* of the original interference pattern $P(\boldsymbol{x})$. Several points now merit comment:

1. The pattern-shift becomes invisible when

$$\varPhi = n\cdot 2\pi\frac{\hbar c}{e} \quad : \quad n = 0, \pm1, \pm2, \ldots \tag{390}$$

This "flux quantization condition" assumes central importance in connection with the physics of superconductors (most notably: that of "superconducting quantum interference devices" or SQUIDs).[225]

2. One sometimes encounters the claim that "The vector potential, though not observable classically, becomes observable in quantum mechanics." The claim is misleading: what becomes quantum mechanically observable is not $\boldsymbol{A}$ itself but the gauge-invariant construct $\oint_{\mathcal{C}}\boldsymbol{A}\cdot d\boldsymbol{x}$, and the element of surprise arises from cases in which $\boldsymbol{B} = \boldsymbol{0}$ everywhere along $\mathcal{C}$. The situation is, however, in some respects quite familiar: at (116) we had

$$\text{Faraday emf} = -\tfrac{1}{c}\tfrac{d}{dt}(\text{enclosed magnetic flux}) = -\tfrac{1}{c}\tfrac{d}{dt}\oint_{\mathcal{C}}\boldsymbol{A}\cdot d\boldsymbol{x}$$

—some engineering applications of which (*e.g.*,, the bevetron) hinge critically on the fact that $\mathcal{C}$ may be remote from the region of changing flux. Here as in the Bohm-Aharanov effect, an element of *non-locality* intrudes.

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[225] See F. Schwabl, *Quantum Mechanics* (2^{nd} edition 1995), §§7.5 & 7.6 or Bjørn Felsager, *Geometry, Particles, and Fields* (1998), §2.12. On the cover of my edition of the latter text, by the way, is a version of my Figure 84, promoted by Felsager to the status of an ikon symbolizing the problem area where geometry/topology and the physics of particles/fields intersect.

3. We found at (386) that

$$\boldsymbol{A} = \boldsymbol{\nabla}\alpha \quad \text{in the region exterior to the solenoid}$$

By gauge transformation $\boldsymbol{A} \longrightarrow \boldsymbol{A} = \boldsymbol{A} - \boldsymbol{\nabla}\alpha$ we construct therefore a vector potential $\boldsymbol{A}$ which *vanishes identically in the exterior region* . . . and presents us with a seeming contradiction:

- We know on the one hand that

$$\oint_{\mathcal{C}} \boldsymbol{A}\cdot d\boldsymbol{x} = \oint_{\mathcal{C}} \boldsymbol{A}\cdot d\boldsymbol{x} \quad \text{by gauge-invariance} \tag{391}$$

- but on the other hand it is clear that

$$\oint_{\mathcal{C}} \boldsymbol{0}\cdot d\boldsymbol{x} = 0$$

Why does this not extinguish the Bohm-Aharonov effect?

The "seeming contradiction" is resolved by the observation that (391) holds if (as is standardly the case) the gauge function is single-valued. But the gauge function α that kills the external solenoidal $\boldsymbol{A}$-field is (see again Figure 86) *multi*-valued, and the contours $\mathcal{C}$ of interest wind from one sheet to the next, so

$$\oint_{\mathcal{C}} \boldsymbol{\nabla}\alpha\cdot d\boldsymbol{x} = \alpha(\text{point}) - \alpha(\text{same point on next-lower sheet}) \neq 0$$

Soon after Michael Berry's discovery (1984) of what came to be called "Berry's phase"—soon recognized to be itself a manifestation of a more general phenomenon called "geometrical phase"—it was pointed out by Aharonov himself that the Bohm-Aharonov effect can be portrayed as a special instance of that deeper and ever more pervasive train of physico-geometrical train of thought . . . that, in short, it represents but the tip of an iceberg.[226]

Conclusion. Potentials are usually considered to enter electrodynamics as mere computational crutches, as aids to simplified formulation of the theory. The same—only more so—can be said of the "superpotentials" of which Hertz gave the first systematic account.[227] We have seen, however, by looking upon Maxwell's theory as a limiting case of Proca's theory . . . that the ghostly status of the potential hangs by a precarious thread: that gauge freedom would be lost, that the potential fields would become directly observable/physical participants in the theory "if only the photon were endowed with mass, however slight."

[226] See Y. Aharonov & J. Anandan, "Phase change in cyclic quantum evolution," PR Letters **58**, 1593 (1987) and other classic papers reprinted in A. Shapere & F. Wilczek, *Geometric Phases in Physics* (1989). Also §10.2.4 in David Griffiths' *Introduction to Quantum Mechanics* (1995).

[227] See §13–4 Wolfgang Panofsky & Melba Phillips, *Classical Electricity & Magnetism* (1955).

With the infusion of quantum mechanical ideas the life of A^μ acquires a dramatic new dimension, and the subject acquires a deeply geometrical flavor. Our review of the Bohm-Aharonov effect has served to illustrate the point, and I have alluded to parallel developments in the theory of superconductivity, but historically prior to either of those is a pretty train of thought set into motion by Dirac in 1931. Dirac[228] put

- the classical electrodynamnics of a magnetic monopole and
- the quantum mechanics of an electrically charged particle

in a bag together. . . shook. . . and came away with an explanation for *why electrical charge is quantized*. We are in position to follow the details only the (very instructive) first part of his argument.[229]

We look (with Dirac) to the vector potential

$$\boldsymbol{A} = (g/4\pi)\begin{pmatrix} \frac{y}{r(r-z)} \\ \frac{-x}{r(r-z)} \\ 0 \end{pmatrix} \quad : \quad r^2 \equiv x^2 + y^2 + z^2 \tag{392}$$

and compute[230]

$$\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A} = (g/4\pi)\,\tfrac{1}{r^3}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{cases} \text{spherically symmetric radial field of} \\ \text{a magnetic monopole of strengh } g \end{cases}$$

. . . as encountered already on page 227. Notice now that on the z-axis (*i.e.*, at $x = y = 0$)

$$\frac{1}{r(r-z)} = \begin{cases} \infty & : \quad z > 0 \\ 1/2z^2 & : \quad z < 0 \end{cases}$$

The potential (392) is called a "Dirac string potential" because it displays a "string singularity" on the positive z-axis. To clarify the mathematical/physical meaning of the singularity we make use once again of the "regularization trick," first encountered on page 12: we write

$$\boldsymbol{A}_\epsilon = (g/4\pi)\begin{pmatrix} \frac{y}{R(R-z)} \\ \frac{-x}{R(R-z)} \\ 0 \end{pmatrix} \quad : \quad R^2 \equiv r^2 + \epsilon^2$$

(from which we recover (392) in the limit $\epsilon \downarrow 0$) and compute

$$\boldsymbol{B}_\epsilon = \boldsymbol{\nabla}\times\boldsymbol{A}_\epsilon = \boldsymbol{B}_\epsilon^{\text{monopole}} + \boldsymbol{B}_\epsilon^{\text{string}}$$

[228] P. A. M. Dirac, Proc. Roy. Soc. London **A133**, 60 (1931); Phys. Rev. **74**, 817 (1948).
[229] For a splendid account of details here omitted see Chapter 9 in Felsager.[225] Also §6.11 in J. D. Jackson, *Classical Electrodynamics* (3rd edition 1999).
[230] PROBLEM 60.

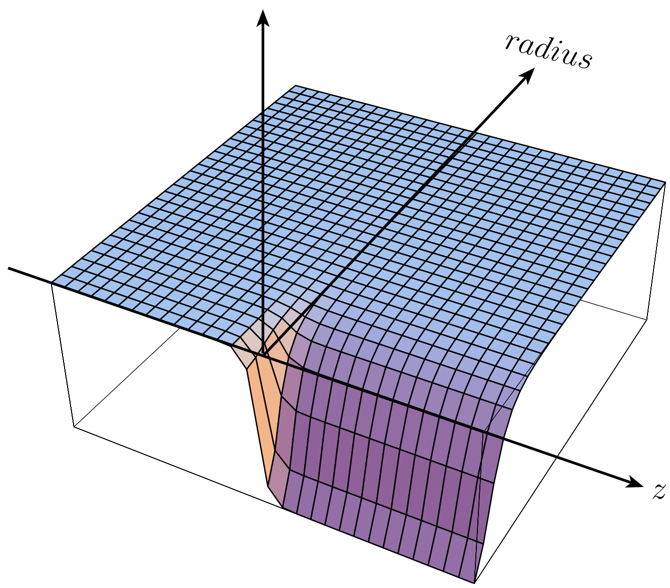

FIGURE 89: $B_\epsilon^{\text{string}}$ *displayed as a function of* z *and*

radius $s \equiv \sqrt{x^2 + y^2}$

The trough along the positive z*-axix gets narrower/deeper as* $\epsilon \downarrow 0$. *The figure refers to the case* $\epsilon = \frac{1}{10}$.

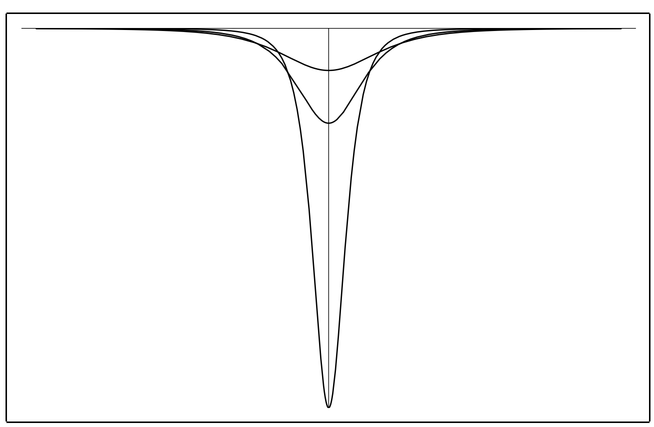

FIGURE 90: *Graphs of the radial dependence of* $B_\epsilon^{\text{string}}$ *at* $z = 1$ *in the cases* $\epsilon = \frac{3}{10}, \frac{2}{10}, \frac{1}{10}$.

with

$$\boldsymbol{B}_\epsilon^{\text{monopole}} = (g/4\pi)\,\frac{1}{R^3}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \boldsymbol{B}_\epsilon^{\text{string}} = (g/4\pi)\begin{pmatrix} 0 \\ 0 \\ B_\epsilon^{\text{string}} \end{pmatrix}$$

where

$$B_\epsilon^{\text{string}} \equiv B^{\text{string}}(z, s; \epsilon) \equiv -\frac{\epsilon^2(2R-z)}{R^3(R-z)^2} \quad : \quad R \equiv \sqrt{s^2+z^2}$$

Clearly

$$\lim_{\epsilon \downarrow 0} \boldsymbol{B}_\epsilon^{\text{monopole}} = \text{monopole field described earlier}$$

It is from the string term, which runs everywhere parallel to the z-axis, that we have things to learn. Figures 89 & 90 tell the story. *Mathematica* informs us that

$$\int_0^\infty B^{\text{string}}(z, s; \epsilon) 2\pi s\, ds = -\frac{2\pi\epsilon^2}{\sqrt{z^2+\epsilon^2}\,(-z+\sqrt{z^2+\epsilon^2}\,)}$$

$$\Big\downarrow \text{limit } \epsilon \downarrow 0$$

$$= \begin{cases} -4\pi & : \quad z > 0 \\ \;\;0 & : \quad z < 0 \end{cases}$$

We are brought thus to the conclusion that $\boldsymbol{B}_\epsilon^{\text{string}}$ is a field such as would arise from a *solenoid of zero cross-section* wrapped around the positive z-axis and carrying a current given by

$$\boldsymbol{j} = \lim_{\epsilon \downarrow 0} \boldsymbol{j}_\epsilon \quad \text{with} \quad \boldsymbol{j}_\epsilon = c\,\boldsymbol{\nabla}\times\boldsymbol{B}_\epsilon^{\text{string}}$$

We learn, moreover, that (see Figure 91)

total magnetic flux delivered down-string by $\boldsymbol{B}^{\text{string}}$
= total magnetic flux delivered spherically outward by $\boldsymbol{B}^{\text{monopole}}$

so the *net* flux through any closed surface containing a Dirac monopole is zero!

One can show that the "string singularity" encountered at (392) is essential, in the sense that it cannot be gauged away. But pretty clearly (and as one can also show), the string can trace *any* curve "from infinity" to the point where it terminates (called "the monopole").

The second part of Dirac's argument is, as already indicated, quantum mechanical: he looks to the quantum motion of an electrically charged particle in the presence of a monopole and stipulates that <u>the string (irrespective of its shape) shall be quantum mechanically invisible</u>. This requirement, which from one point of view serves to fix the pitch of the multivalued superpotential

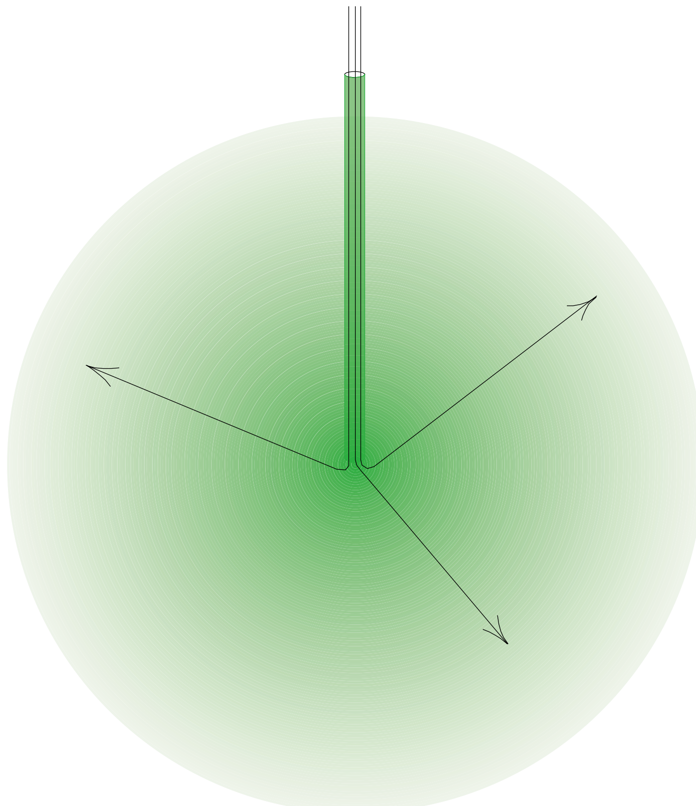

FIGURE 91: *Magnetic field and field lines of the Dirac monopole described in the text. The net magnetic flux through any surface that encloses the monopole is zero. Dirac's idealized "string solenoid" is shown (here as in the text) to be coincident with the positive z-axis, but can in general trace any curve from the location of the monopole "to infinity." My use of the phrase "from the monopole" is perhaps misleading: for Dirac the monopole is the dangling free end of the string solenoid.*

(Figure 86), can be phrased as a requirement that the string give rise to a null Bohm-Aharonov effect (this 25 years before the ostensible *discovery* of the Bohm-Aharonov effect!). One is led thus from (390) to the **Dirac quantization condition**

$$\text{string flux } g = n \cdot 2\pi\frac{\hbar c}{e}$$

This is precisely the condition

$$\text{angular momentum of Tompson's mixed dipole } \frac{eg}{4\pi c} = n \cdot \tfrac{1}{2}\hbar$$

to which we were led on page 232 by quite another (and less compelling) line of argument. The strongest conclusion that can be drawn from either argument is that the product $e{\cdot}g$ is quantized:

$$eg = n \cdot 2\pi\hbar c$$

A fundamentally new idea would be required to account theoretically for this observed fact of Nature:

e—and therefore also g—are *individually* quantized

Dirac's argument does not quite do the job; to pretend otherwise (a common practice) is to engage in some wishful thinking . . . and to decline an invitation to invention.

The Bohm-Aharonov effect and its siblings—seen now to include flux and charge quantization—are topological children of a liaison between A^μ and quantum mechanics. Gauge field theory is, if anything, even more deeply geometrical. Drawing covertly upon ideas (covariant differentiation, curvature) borrowed from differential geometry, it portrays electrodynamics as "the price one pays" in order to promote the global phase invariance

$$\psi \longrightarrow \psi = \quad e^{i\,g\chi} \cdot \psi \quad : \quad \chi \text{ any real constant}$$

standard to quantum theory . . . to an invariance with respect to *local* phase transformations

$$\psi \longrightarrow \psi = e^{i\,g\chi(x)} \cdot \psi \quad : \quad \chi(x) \text{ any real-valued function of } x$$

This is accomplished by in effect pursuing in reverse the argument which on pages 273–274 was used to establish the electrodynamical gauge-invariance of quantum mechanics: we adjust the meaning (of momentum; *i.e.*, of) the differentiation operator

$$\partial_\mu \longrightarrow \mathcal{D}_\mu \equiv \partial_\mu - igA_\mu$$

and achieve the desired local phase (or gauge) invariance by stipulating that the "compensating field" A_μ will participate in the transformation by the rule (383). Finally (by a mechanism natural to Lagrangian field theory) we launch the compensating field into motion and find that it satisfies precisely the equation

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = \tfrac{1}{c}j^\nu$$

that at (371) was found to comprise "Maxwell's theory in a nutshell." The theory leads, moreover, to an explicit description of the current 4-vector j_μ. Directly observable "physicality" is assigned—from a formal point of view almost as an afterthought!—to the gauge-invariant construction

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \quad : \quad \text{analog of geometrical "curvature"}$$

No mere crutch, A_μ has by this point become arguably the *principal object* in Maxwellian electrodynamics—the wellspring from which the theory flows. And in quantum electrodynamics (QED) it is, moreover, A_μ—not $F_{\mu\nu}$ but A_μ—that is "quantized."[231]

For several decades the program just described was dismissed as a formal curiosity, an exercise that led to nothing not already known. But in the 1950's it was discovered (by Yank & Mills, Shaw, Umazawa[211]) that it admits readily and elegantly of profound generalization, that it can be used to construct Maxwell-like theories of the non-electromagnetic interactions among elementary particles—"non-Abelian gauge theories" that appear to be in generally excellent agreement with the observational facts.[232] Physics provides no more persuasive evidence that Truth and stunning Beauty come often to the same thing.

It may be fair, as I did at the outset, to refer to potentials (and, more generally, to gauge fields) as "spooks," as sirens who discretely hide their nakedness, but such language leaves half the story untold: they are spooks who spring from the deepest darkest places, who come to us murmuring of the most obscure symmetries of Nature . . . and who appear to be in formal control of Reality.

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[231] See, for example, J. M. Jauch & F. Rohrlich, *The Theory of Photons & Electrons* (1955), §2–4.

[232] For an elementary introduction to this inexhaustibly rich subject see, for example, the final Chapter 11 in David Griffiths' *Introduction to Elementary Particles* (1987).